NOTE ON MATH 4010: FUNCTIONAL ANALYSIS

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Throughout this note, all spaces $X, Y, ..$ are normed spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $B_X := \{x \in X : \|x\| \leq 1\}$ and $S_X := \{x \in X : \|x\| = 1\}$ denote the closed unit ball and the unit sphere of X respectively.

1. Classical Normed spaces

Proposition 1.1. *Let X be a normed space. Then the following assertions are equivalent.*

(i) X is a Banach space.

(ii) If a series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent in X, i.e., $\sum_{n=1}^{\infty} \|x_n\| < \infty$, implies that the series $\sum_{n=1}^{\infty} x_n$ converges in the norm.

Proof. $(i) \Rightarrow (ii)$ is obvious.
Now suppose that Part (ii) holds. Let (y_n) be a Cauchy sequence in X. It suffices to show that (y_n) has a convergent subsequence. In fact, by the definition of a Cauchy sequence, there is a subsequence (y_{n_k}) such that $\|y_{n_{k+1}} - y_{n_k}\| < \frac{1}{2^k}$ for all $k = 1, 2....$ So by the assumption, the series $\sum_{k=1}^{\infty}(y_{n_{k+1}} - y_{n_k})$ converges in the norm and hence, the sequence (y_{n_k}) is convergent in X. The proof is finished. □

Throughout the note, we write a sequence of numbers as a function $x : \{1, 2, ...\} \to \mathbb{K}$.
The following examples are important classes in the study of functional analysis.

Example 1.2. *Put*

$$c_0 := \{(x(i)) : x(i) \in \mathbb{K}, \ \lim |x(i)| = 0\} \text{ and } \ell^{\infty} := \{(x(i)) : x(i) \in \mathbb{K}, \ \sup_i x(i) < \infty\}.$$

Then c_0 is a subspace of ℓ^{∞}. The sup*-norm $\|\cdot\|_{\infty}$ on ℓ^{∞} is defined by $\|x\|_{\infty} := \sup_i |x(i)|$ for $x \in \ell^{\infty}$. Then ℓ^{∞} is a Banach space and $(c_0, \|\cdot\|_{\infty})$ is a closed subspace of ℓ^{∞} (**Check !**) and hence c_0 is also a Banach space too.*
Let

$$c_{00} := \{(x(i)) : \text{ there are only finitly many } x(i)\text{'s are non-zero}\}.$$

Also, c_{00} is endowed with the sup*-norm defined above. Then c_{00} is not a Banach space (**Why?**) but it is dense in c_0, that is, $\overline{c_{00}} = c_0$ (**Check!**).*

Example 1.3. *For $1 \leq p < \infty$. Put*

$$\ell^p := \{(x(i)) : x(i) \in \mathbb{K}, \ \sum_{i=1}^{\infty} |x(i)|^p < \infty\}.$$

Also, ℓ^p is equipped with the norm $\|x\|_p := (\sum_{i=1}^{\infty} |x(i)|^p)^{\frac{1}{p}}$ for $x \in \ell^p$. Then ℓ^p becomes a Banach space under the norm $\|\cdot\|_p$.

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Example 1.4. *Let X be a locally compact Hausdorff space, for example, $\mathbb{K}$. Let $C_0(X)$ be the space of all continuous $\mathbb{K}$-valued functions f on X which are vanish at infinity, that is, for every $\varepsilon > 0$, there is a compact subset D of X such that $|f(x)| < \varepsilon$ for all $x \in X \setminus D$. Now $C_0(X)$ is endowed with the* sup*-norm, that is,*

$$\|f\|_\infty = \sup_{x\in X} |f(x)|$$

*for every $f \in C_0(X)$. Then $C_0(X)$ is a Banach space. (***Try to prove this fact for the case $X = \mathbb{R}$. Just use the knowledge from MATH 2060 !!!***)*

2. Finite Dimensional Normed Spaces

We say that two norms $\|\cdot\|$ and $\|\cdot\|'$ on a vector space X are *equivalent*, write $\|\cdot\| \sim \|\cdot\|'$, if there are positive numbers c_1 and c_2 such that $c_1\|\cdot\| \leq \|\cdot\|' \leq c_2\|\cdot\|$ on X.

Example 2.1. *Consider the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on ℓ^1. We are going to show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not equivalent. In fact, if we put $x_n(i) := (1, 1/2, ..., 1/n, 0, 0, ...)$ for $n, i = 1, 2....$ Then $x_n \in \ell^1$ for all n. Notice that (x_n) is a Cauchy sequence with respect to the norm $\|\cdot\|_\infty$ but it is not a Cauchy sequence with respect to the norm $\|\cdot\|_1$. Hence $\|\cdot\|_1 \nsim \|\cdot\|_\infty$ on ℓ^1.*

Proposition 2.2. *All norms on a finite dimensional vector space are equivalent.*

Proof. Let X be a finite dimensional vector space and let $\{e_1, ..., e_n\}$ be a vector base of X. For each $x = \sum_{i=1}^n \alpha_i e_i$ for $\alpha_i \in \mathbb{K}$, define $\|x\|_0 = \sum_{i=1}^n |\alpha_i|$. Then $\|\cdot\|_0$ is a norm X. The result is obtained by showing that all norms $\|\cdot\|$ on X are equivalent to $\|\cdot\|_0$.
Notice that for each $x = \sum_{i=1}^n \alpha_i e_i \in X$, we have $\|x\| \leq (\max\limits_{1\leq i\leq n} \|e_i\|)\|x\|_0$. It remains to find $c > 0$ such that $c\|\cdot\|_0 \leq \|\cdot\|$. In fact, let $\mathbb{K}^n$ be equipped with the sup-norm $\|\cdot\|_\infty$, that is $\|(\alpha_1, ..., \alpha_n)\|_\infty = \max_{1\leq 1\leq n} |\alpha_i|$. Define a real-valued function f on the unit sphere $S_{\mathbb{K}^n}$ of $\mathbb{K}^n$ by

$$f : (\alpha_1, ..., \alpha_n) \in S_{\mathbb{K}^n} \mapsto \|\alpha_1 e_1 + \cdots + \alpha_n e_n\|.$$

Notice that the map f is continuous and $f > 0$. It is clear that $S_{\mathbb{K}^n}$ is compact with respect to the sup-norm $\|\cdot\|_\infty$ on $\mathbb{K}^n$. Hence, there is $c > 0$ such that $f(\alpha) \geq c > 0$ for all $\alpha \in S_{\mathbb{K}^n}$. This gives $\|x\| \geq c\|x\|_0$ for all $x \in X$ as desired. The proof is finished. □

Corollary 2.3. *We have the following assertions.*

(i) All finite dimensional normed spaces are Banach spaces. Consequently, any finite dimensional subspace of a normed space must be closed.
(ii) The closed unit ball of any finite dimensional normed space is compact.

Proof. Let $(X, \|\cdot\|)$ be a finite dimensional normed space. With the notation as in the proof of Proposition 2.2 above, we see that $\|\cdot\|$ must be equivalent to the norm $\|\cdot\|_0$. It is clear that X is complete with respect to the norm $\|\cdot\|_0$ and so is complete in the original norm $\|\cdot\|$. The Part (i) follows.
For Part (ii), it is clear that the compactness of the closed unit ball of X is equivalent to saying that any closed and bounded subset being compact. Therefore, Part (ii) follows from the simple observation that any closed and bounded subset of X with respect to the norm $\|\cdot\|_0$ is compact. The proof is complete. □

In the rest of this section, we are going to show the converse of Corollary 2.3(ii) also holds. Before this result, we need the following useful result.

Lemma 2.4. Riesz's Lemma: *Let Y be a closed proper subspace of a normed space X. Then for each $\theta \in (0, 1)$, there is an element $x_0 \in S_X$ such that $d(x_0, Y) := \inf\{\|x_0 - y\| : y \in Y\} \geq \theta$.*

Proof. Let $u \in X - Y$ and $d := \inf\{\|u - y\| : y \in Y\}$. Notice that since Y is closed, $d > 0$ and hence, we have $0 < d < \frac{d}{\theta}$ because $0 < \theta < 1$. This implies that there is $y_0 \in Y$ such that $0 < d \le \|u - y_0\| < \frac{d}{\theta}$. Now put $x_0 := \frac{u - y_0}{\|u - y_0\|} \in S_X$. We are going to show that x_0 is as desired. Indeed, let $y \in Y$. Since $y_0 + \|u - y_0\|y \in Y$, we have

$$\|x_0 - y\| = \frac{1}{\|u - y_0\|}\|u - (y_0 + \|u - y_0\|y)\| \ge d/\|u - y_0\| > \theta.$$

So, $d(x_0, Y) \ge \theta$. □

Remark 2.5. The Riesz's lemma does not hold when $\theta = 1$. The following example can be found in the Diestel's interesting book without proof (see [2, Chapter 1 Ex.3(i)]).
Let $X = \{x \in C([0,1], \mathbb{R}) : x(0) = 0\}$ and $Y = \{y \in X : \int_0^1 y(t)dt = 0\}$. Both X and Y are endowed with the sup-norm. Notice that Y is a closed proper subspace of X. We are going to show that for any $x \in S_X$, there is $y \in Y$ such that $\|x - y\|_\infty < 1$. Thus, the Riesz's Lemma does not hold as $\theta = 1$ in this case.
In fact, let $x \in S_X$. Since $x(0) = 0$ with $\|x\|_\infty = 1$, we can find $0 < a < 1/4$ such that $|x(t)| \le 1/4$ for all $t \in [0, a]$. Notice that since x is uniform continuous on $[a, 1]$, for any $0 < \varepsilon < 1/4$, there is $\delta > 0$ such that $|x(t) - x(t')| < \varepsilon/4$ when $|t - t'| < \delta$. Now we find a partition $a = t_0 < t_1 < \cdots < t_n = 1$ with $t_k - t_{k-1} < \delta$ for all $k = 1, 2, ...n$ and $|x(t_k)| < 1$ for all $k = 1, 2..., n-1$. Then $\sup\{|x(t) - x(t')| : t, t' \in [t_{k-1}, t_k]\} < \varepsilon/4$. We let $p_{k-1} := \sup\{t \in [t_{k-1}, t_k] : x|_{[t_{k-1},t]} > -1 + \varepsilon\}$ if it exists, otherwise, put $p_{k-1} := t_{k-1}$. Similarly, let $q_k := \inf\{t \in [t_{k-1}, t_k] : x|_{[t,t_k]} > -1 + \varepsilon\}$ if it exists, otherwise, put $q_k := t_k$. So, one can find a continuous function ϕ on $[a, 1]$ such that

$$\phi(t) = \begin{cases} \varepsilon & \text{if } t \in [t_{k-1}, t_k] \text{ and } x|_{[t_{k-1},t_k]} > -1 + \varepsilon. \\ -\varepsilon & \text{if } t \in [p_{k-1}, q_k] \text{ and } x|_{[t_{k-1},t_k]} \not> -1 + \varepsilon. \\ \frac{-2\varepsilon}{p_{k-1} - t_{k-1}}(t - t_{k-1}) + \varepsilon & \text{if } x|_{[t_{k-1},t_k]} \not> -1 + \varepsilon \text{ and } t_{k-1} < t < p_{k-1}. \\ \frac{2\varepsilon}{t_k - q_k}(t - t_k) + \varepsilon & \text{if } x|_{[t_{k-1},t_k]} \not> -1 + \varepsilon \text{ and } q_k < t < t_k. \end{cases}$$

Notice that if $x|_{[t_{k-1},t_k]} \not> -1 + \varepsilon$, then $t_{k-1} < p_{k-1}$ or $q_k < t_k$. So, $\|x|_{[a,1]} - \phi\|_\infty < 1$.
It is because $\|\phi\|_\infty < 2\varepsilon$, we have $|\int_a^1 \phi(t)dt| \le 2\varepsilon(1 - a)$. On the other hand, as $|x(t)| < 1/4$ on $[0, a]$, so if we further choose ε small enough such that $(1 - a)(2\varepsilon) < a/4$, then we can find a continuous function y_1 on $[0, a]$ such that $|y_1(t)| < 1/4$ on $[0, a]$ with $; y_1(0) = 0; y_1(a) = x(a)$ and $\int_0^a y_1(t)dt = -\int_a^1 \phi(t)dt$. Now we define $y = y_1$ on $[0, a]$ and $y = \phi$ on $[a, 1]$. Then $\|y - x\|_\infty < 1$ and $y \in Y$ is as desired.

Theorem 2.6. *X is a finite dimensional normed space if and only if the closed unit ball B_X of X is compact.*

Proof. The necessary condition has been shown by Proposition 2.3(ii).
Now assume that X is of infinite dimension. Fix an element $x_1 \in S_X$. Let $Y_1 = \mathbb{K}x_1$. Then Y_1 is a proper closed subspace of X. The Riesz's lemma gives an element $x_2 \in S_X$ such that $\|x_1 - x_2\| \ge 1/2$. Now consider $Y_2 = span\{x_1, x_2\}$. Then Y_2 is a proper closed subspace of X since $\dim X = \infty$. To apply the Riesz's Lemma again, there is $x_3 \in S_X$ such that $\|x_3 - x_k\| \ge 1/2$ for $k = 1, 2$. To repeat the same step, there is a sequence $(x_n) \in S_X$ such that $\|x_m - x_n\| \ge 1/2$ for all $n \ne m$. Thus, (x_n) is a bounded sequence without any convergence subsequence. So, B_X is not compact. The proof is finished. □

Recall that a metric space Z is said to be *locally compact* if for any point $z \in Z$, there is a compact neighborhood of z. Theorem 2.6 implies the following corollary immediately.

Corollary 2.7. *Let X be a normed space. Then X is locally compact if and only if $\dim X < \infty$.*

3. Bounded Linear Operators

Proposition 3.1. *Let T be a linear operator from a normed space X into a normed space Y. Then the following statements are equivalent.*

(i) T *is continuous on* X*.*
(ii) T *is continuous at* $0 \in X$*.*
(iii) $\sup\{\|Tx\| : x \in B_X\} < \infty$.

In this case, let $\|T\| = \sup\{\|Tx\| : x \in B_X\}$ *and* T *is said to be bounded.*

Proof. $(i) \Rightarrow (ii)$ is obvious.
For $(ii) \Rightarrow (i)$, suppose that T is continuous at 0. Let $x_0 \in X$. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that $\|Tw\| < \varepsilon$ for all $w \in X$ with $\|w\| < \delta$. Therefore, we have $\|Tx - Tx_0\| = \|T(x - x_0)\| < \varepsilon$ for any $x \in X$ with $\|x - x_0\| < \delta$. So, (i) follows.
For $(ii) \Rightarrow (iii)$, since T is continuous at 0, there is $\delta > 0$ such that $\|Tx\| < 1$ for any $x \in X$ with $\|x\| < \delta$. Now for any $x \in B_X$ with $x \neq 0$, we have $\|\frac{\delta}{2}x\| < \delta$. So, we see have $\|T(\frac{\delta}{2}x)\| < 1$ and hence, we have $\|Tx\| < 2/\delta$. So, (iii) follows.
Finally, it remains to show $(iii) \Rightarrow (ii)$. Notice that by the assumption of (iii), there is $M > 0$ such that $\|Tx\| \leq M$ for all $x \in B_X$. So, for each $x \in X$, we have $\|Tx\| \leq M\|x\|$. This implies that T is continuous at 0. The proof is complete. □

Corollary 3.2. *Let $T : X \to Y$ be a bounded linear map. Then we have*

$$\sup\{\|Tx\| : x \in B_X\} = \sup\{\|Tx\| : x \in S_X\} = \inf\{M > 0 : \|Tx\| \leq M\|x\|,\ \forall x \in X\}.$$

Proof. Let $a = \sup\{\|Tx\| : x \in B_X\}$, $b = \sup\{\|Tx\| : x \in S_X\}$ and $c = \inf\{M > 0 : \|Tx\| \leq M\|x\|,\ \forall x \in X\}$.
It is clear that $b \leq a$. Now for each $x \in B_X$ with $x \neq 0$, then we have $b \geq \|T(x/\|x\|)\| = (1/\|x\|)\|Tx\| \geq \|Tx\|$. So, we have $b \geq a$ and thus, $a = b$.
Now if $M > 0$ satisfies $\|Tx\| \leq M\|x\|$, $\forall x \in X$, then we have $\|Tw\| \leq M$ for all $w \in S_X$. So, we have $b \leq M$ for all such M. So, we have $b \leq c$. Finally, it remains to show $c \leq b$. Notice that by the definition of b, we have $\|Tx\| \leq b\|x\|$ for all $x \in X$. So, $c \leq b$. □

Proposition 3.3. *If X is of finite dimension normed space, then for any linear operator T from X into a normed space Y must be bounded.*

Proof. Let $\|\cdot\|_0$ be the equivalent norm on X defined as in the proof of Proposition 2.2. It is clear that T is continuous at 0 with respect to the norm $\|\cdot\|_0$. So, T is bounded by Proposition 3.1 at once. □

Proposition 3.4. *Let Y be a closed subspace of X and X/Y be the quotient space. For each element $x \in X$, put $\bar{x} := x + Y \in X/Y$ the corresponding element in X/Y. Define*

$$\|\bar{x}\| = \inf\{\|x + y\| : y \in Y\}. \tag{3.1}$$

If we let $\pi : X \to X/Y$ be the natural projection, that is $\pi(x) = \bar{x}$ for all $x \in X$, then $(X/Y, \|\cdot\|)$ is a normed space and π is bounded with $\|\pi\| \leq 1$. In particular, $\|\pi\| = 1$ as Y is a proper closed subspace.
Furthermore, if X is a Banach space, then so is X/Y.
In this case, we call $\|\cdot\|$ in (3.1) the quotient norm on X/Y.

Proof. Notice that since Y is closed, one can directly check that $\|\bar{x}\| = 0$ if and only is $x \in Y$, that is, $\bar{x} = \bar{0} \in X/Y$. It is easy to check the other conditions of the definition of a norm. So, X/Y is a normed space. Also, it is clear that π is bounded with $\|\pi\| \leq 1$ by the definition of the quotient norm on X/Y.

Furthermore, if $Y \subsetneq X$, then by using the Riesz's Lemma 2.4, we see that $\|\pi\| = 1$ at once.
We are going to show the last assertion. Suppose that X is a Banach space. Let $(\bar{x}_n)$ be a Cauchy sequence in X/Y. It suffices to show that $(\bar{x}_n)$ has a convergent subsequence in X/Y (**Why?**). Indeed, since $(\bar{x}_n)$ is a Cauchy sequence, we can find a subsequence $(\bar{x}_{n_k})$ of $(\bar{x}_n)$ such that

$$\|\bar{x}_{n_{k+1}} - \bar{x}_{n_k}\| < 1/2^k$$

for all $k = 1, 2....$ Then by the definition of quotient norm, there is an element $y_1 \in Y$ such that $\|x_{n_2} - x_{n_1} + y_1\| < 1/2$. Notice that we have, $\overline{x_{n_1} - y_1} = \bar{x}_{n_1}$ in X/Y. So, there is $y_2 \in Y$ such that $\|x_{n_2} - y_2 - (x_{n_1} - y_1)\| < 1/2$ by the definition of quotient norm again. Also, we have $\overline{x_{n_2} - y_2} = \bar{x}_{n_2}$. Then we also have an element $y_3 \in Y$ such that $\|x_{n_3} - y_3 - (x_{n_2} - y_2)\| < 1/2^2$. To repeat the same step, we can obtain a sequence (y_k) in Y such that

$$\|x_{n_{k+1}} - y_{k+1} - (x_{n_k} - y_k)\| < 1/2^k$$

for all $k = 1, 2....$ Therefore, $(x_{n_k} - y_k)$ is a Cauchy sequence in X and thus, $\lim_k (x_{n_k} - y_k)$ exists in X while X is a Banach space. Set $x = \lim_k (x_{n_k} - y_k)$. On the other hand, notice that we have $\pi(x_{n_k} - y_k) = \pi(x_{n_k})$ for all $k = 1, 2, , ,$. This tells us that $\lim_k \pi(x_{n_k}) = \lim_k \pi(x_{n_k} - y_k) = \pi(x) \in X/Y$ since π is bounded. So, $(\bar{x}_{n_k})$ is a convergent subsequence of $(\bar{x}_n)$ in X/Y. The proof is complete. □

Corollary 3.5. *Let $T : X \to Y$ be a linear map. Suppose that Y is of finite dimension. Then T is bounded if and only if $\ker T := \{x \in X : Tx = 0\}$, the kernel of T, is closed.*

Proof. The necessary part is clear.
Now assume that $\ker T$ is closed. Then by Proposition 3.4, $X/\ker T$ becomes a normed space. Also, it is known that there is a linear injection $\widetilde{T} : X/\ker T \to Y$ such that $T = \widetilde{T} \circ \pi$, where $\pi : X \to X/\ker T$ is the natural projection. Since $\dim Y < \infty$ and $\widetilde{T}$ is injective, $\dim X/\ker T < \infty$. This implies that $\widetilde{T}$ is bounded by Proposition 3.3. Hence T is bounded because $T = \widetilde{T} \circ \pi$ and π is bounded. □

Remark 3.6. The converse of Corollary 3.5 does not hold when Y is of infinite dimension. For example, let $X := \{x \in \ell^2 : \sum_{n=1}^{\infty} n^2 |x(n)|^2 < \infty\}$ (notice that X is a vector space **Why?**) and $Y = \ell^2$. Both X and Y are endowed with $\|\cdot\|_2$-norm.
Define $T : X \to Y$ by $Tx(n) = nx(n)$ for $x \in X$ and $n = 1, 2....$ Then T is an unbounded operator(**Check !!**). Notice that $\ker T = \{0\}$ and hence, $\ker T$ is closed. So, the closeness of $\ker T$ does not imply the boundedness of T in general.

We say that two normed spaces X and Y are said to be *isomorphic (resp. isometric isomorphic)* if there is a bi-continuous linear isomorphism (resp. isometric) between X and Y. We also write $X = Y$ if X and Y are isometric isomorphic.

Recall that a metric space is said to be *separable* if there is a countable dense subset, for example, the base field $\mathbb{K}$ is separable. Also, it is easy to see that the separability is preserved under a homeomorphism.

Definition 3.7. *We say that a sequence of element $(e_n)_{n=1}^{\infty}$ in a normed space X is called a Schauder base for X if for each element $x \in X$, there is a unique sequence of scalars (α_n) such that*

$$x = \sum_{n=1}^{\infty} \alpha_n e_n. \tag{3.2}$$

Note: The expression in Eq. 3.2 depends on the order of e_n's.

Remark 3.8. Notice that if X has a Scahuder base, then X must be separable. The following natural question we first raised by Banach (1932).
The base problem: Does every separable Banach space have a Schauder base?
The answer is "**No**″!
This problem was completely solved by P. Enflo in 1973.

Example 3.9. *We have the following assertions.*

(i) The space ℓ^∞ is non-separable under the sup*-norm $\|\cdot\|_\infty$. Consequently, ℓ^∞ has no Schauder base.*
(ii) The spaces c_0 and ℓ^p for $1 \leq p < \infty$ have Schauder bases.

Proof. For Part (i) let $D = \{x \in \ell^\infty : x(i) = 0 \text{ or } 1\}$. Then D is an uncountable set and $\|x - y\|_\infty = 1$ for $x \neq y$. Therefore $\{B(x, 1/4) : x \in D\}$ is an uncountable family of disjoint open balls. So, ℓ^∞ has no countable dense subset.
For each $n = 1, 2...$, let $e_n(i) = 1$ if $n = i$, otherwise, is equal to 0.
Also, (e_n) is a Schauder base for the space c_0 and ℓ^p for $1 \leq p < \infty$. □

Proposition 3.10. *Let X and Y be normed spaces. Let $B(X, Y)$ be the set of all bounded linear maps from X into Y. For each element $T \in B(X, Y)$, let*

$$\|T\| = \sup\{\|Tx\| : x \in B_X\}.$$

be defined as in Proposition 3.1.
Then $(B(X, Y), \|\cdot\|)$ becomes a normed space.
Furthermore, if Y is a Banach space, then so is $B(X, Y)$.

Proof. One can directly check that $B(X, Y)$ is a normed space (**Do It By Yourself!**).
We are going to show that $B(X, Y)$ is complete if Y is a Banach space. Let (T_n) be a Cauchy sequence in $L(X, Y)$. Then for each $x \in X$, it is easy to see that $(T_n x)$ is also a Cauchy sequence in Y. So, $\lim T_n x$ exists in Y for each $x \in X$ because Y is complete. Hence, one can define a map $Tx := \lim T_n x \in Y$ for each $x \in X$. It is clear that T is a linear map from X into Y.
It needs to show that $T \in L(X, Y)$ and $\|T - T_n\| \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. Since (T_n) is a Cauchy sequence in $L(X, Y)$, there is a positive integer N such that $\|T_m - T_n\| < \varepsilon$ for all $m, n \geq N$. So, we have $\|(T_m - T_n)(x)\| < \varepsilon$ for all $x \in B_X$ and $m, n \geq N$. Taking $m \to \infty$, we have $\|Tx - T_n x\| \leq \varepsilon$ for all $n \geq N$ and $x \in B_X$. Therefore, we have $\|T - T_n\| \leq \varepsilon$ for all $n \geq N$. From this, we see that $T - T_N \in B(X, Y)$ and thus, $T = T_N + (T - T_N) \in B(X, Y)$ and $\|T - T_n\| \to 0$ as $n \to \infty$. Therefore, $\lim_n T_n = T$ exists in $B(X, Y)$. □

4. Dual Spaces

By Proposition 3.10, we have the following assertion at once.

Proposition 4.1. *Let X be a normed space. Put $X^* = B(X, \mathbb{K})$. Then X^* is a Banach space and is called the dual space of X.*

Example 4.2. *Let $X = \mathbb{K}^N$. Consider the usual Euclidean norm on X, that is, $\|(x_1, ..., x_N)\| := \sqrt{|x_1|^2 + \cdots |x_N|^2}$. Define $\theta : \mathbb{K}^N \to (\mathbb{K}^N)^*$ by $\theta x(y) = x_1 y_1 + \cdots + x_N y_N$ for $x = (x_1, ..., x_N)$ and $y = (y_1, ..., y_N) \in \mathbb{K}^N$. Notice that $\theta x(y) = \langle x, y \rangle$, the usual inner product on $\mathbb{K}^N$. Then by the Cauchy-Schwarz inequality, it is easy to see that θ is an isometric isomorphism. Therefore, we have $\mathbb{K}^N = (\mathbb{K}^N)^*$.*

Example 4.3. *Define a map* $T : \ell^1 \to c_0^*$ *by*

$$(Tx)(\eta) = \sum_{i=1}^{\infty} x(i)\eta(i)$$

for $x \in \ell^1$ *and* $\eta \in c_0$.
Then T *is isometric isomorphism and hence,* $c_0^* = \ell^1$.

Proof. The proof is divided into the following steps.
Step 1. $Tx \in c_0^*$ for all $x \in \ell^1$.
In fact, let $\eta \in c_0$. Then

$$|Tx(\eta)| \leq |\sum_{i=1}^{\infty} x(i)\eta(i)| \leq \sum_{i=1}^{\infty} |x(i)||\eta(i)| \leq \|x\|_1\|\eta\|_\infty.$$

So, *Step* 1 follows.
Step 2. T is an isometry.
Notice that by *Step* 1, we have $\|Tx\| \leq \|x\|_1$ for all $x \in \ell^1$. It needs to show that $\|Tx\| \geq \|x\|_1$ for all $x \in \ell^1$. Fix $x \in \ell^1$. Now for each $k = 1, 2..$, consider the polar form $x(k) = |x(k)|e^{i\theta_k}$. Notice that $\eta_n := (e^{-i\theta_1}, ..., e^{-i\theta_n}, 0, 0, ...) \in c_0$ for all $n = 1, 2....$ Then we have

$$\sum_{k=1}^{n} |x(k)| = \sum_{k=1}^{n} x(k)\eta_n(k) = Tx(\eta_n) = |Tx(\eta_n)| \leq \|Tx\|$$

for all $n = 1, 2....$ So, we have $\|x\|_1 \leq \|Tx\|$.
Step 3. T is a surjection.
Let $\phi \in c_0^*$ and let $e_k \in c_0$ be given by $e_k(j) = 1$ if $j = k$, otherwise, is equal to 0. Put $x(k) := \phi(e_k)$ for $k = 1, 2...$ and consider the polar form $x(k) = |x(k)|e^{i\theta_k}$ as above. Then we have

$$\sum_{k=1}^{n} |x(k)| = \phi(\sum_{k=1}^{n} e^{-i\theta_k} e_k) \leq \|\phi\|\|\sum_{k=1}^{n} e^{-i\theta_k} e_k\|_\infty = \|\phi\|$$

for all $n = 1, 2....$ Therefore, $x \in \ell^1$.
Finally, we need to show that $Tx = \phi$ and thus, T is surjective. In fact, if $\eta = \sum_{k=1}^{\infty} \eta(k)e_k \in c_0$, then we have

$$\phi(\eta) = \sum_{k=1}^{\infty} \eta(k)\phi(e_k) = \sum_{k=1}^{\infty} \eta(k)x_k = Tx(\eta).$$

So, the proof is finished by the *Steps* 1-3 above. □

Example 4.4. We have the other important examples of the dual spaces.

(i) $(\ell^1)^* = \ell^\infty$.
(ii) For $1 < p < \infty$, $(\ell^p)^* = \ell^q$, where $\frac{1}{p} + \frac{1}{q} = 1$.
(iii) For a locally compact Hausdorff space X, $C_0(X)^* = M(X)$, where $M(X)$ denotes the space of all regular Borel measures on X.

Parts (i) and (ii) can be obtained by the similar argument as in Example 4.3 (see also in [3, Chapter 8]). Part (iii) is known as the *Riesz representation Theorem* which is referred to [3, Section 21.5] for the details.

5. Hahn-Banach Theorem

All spaces $X, Y, Z...$ are normed spaces over the field $\mathbb{K}$ throughout this section.

Lemma 5.1. *Let Y be a subspace of X and $v \in X \setminus Y$. Let $Z = Y \oplus \mathbb{K}v$ be the linear span of Y and v in X. If $f \in Y^*$, then there is an extension $F \in Z^*$ of f such that $\|F\| = \|f\|$.*

Proof. We may assume that $\|f\| = 1$ by considering the normalization $f/\|f\|$ if $f \neq 0$.
Case $\mathbb{K} = \mathbb{R}$:
We first note that since $\|f\| = 1$, we have $|f(x) - f(y)| \leq \|(x+v) - (y+v)\|$ for all $x, y \in Y$. This implies that $-f(x) - \|x+v\| \leq -f(y) + \|y+v\|$ for all $x, y \in Y$. Now let $\gamma = \sup\{-f(x) - \|x+v\| : x \in X\}$. This implies that γ exists and

$$-f(y) - \|y+v\| \leq \gamma \leq -f(y) + \|y+v\| \tag{5.1}$$

for all $y \in Y$. We define $F : Z \longrightarrow \mathbb{R}$ by $F(y + \alpha v) := f(y) + \alpha\gamma$. It is clear that $F|_Y = f$. For showing $F \in Z^*$ with $\|F\| = 1$, since $F|_Y = f$ on Y and $\|f\| = 1$, it needs to show $|F(y + \alpha v)| \leq \|y + \alpha v\|$ for all $y \in Y$ and $\alpha \in \mathbb{R}$.
In fact, for $y \in Y$ and $\alpha > 0$, then by inequality 5.1, we have

$$|F(y + \alpha v)| = |f(y) + \alpha\gamma| \leq \|y + \alpha v\|. \tag{5.2}$$

Since y and α are arbitrary in inequality 5.2, we see that $|F(y + \alpha v)| \leq \|y + \alpha v\|$ for all $y \in Y$ and $\alpha \in \mathbb{R}$. Therefore the result holds when $\mathbb{K} = \mathbb{R}$.
Now for the complex case, let $h = \Re e f$ and $g = \Im m f$. Then $f = h + ig$ and f, g both are real linear with $\|h\| \leq 1$. Note that since $f(iy) = if(y)$ for all $y \in Y$, we have $g(y) = -h(iy)$ for all $y \in Y$. This gives $f(\cdot) = h(\cdot) - ih(i\cdot)$ on Y. Then by the real case above, there is a real linear extension H on $Z := Y \oplus \mathbb{R}v \oplus i\mathbb{R}v$ of h such that $\|H\| = \|h\|$. Now define $F : Z \longrightarrow \mathbb{C}$ by $F(\cdot) := H(\cdot) - iH(i\cdot)$. Then $F \in Z^*$ and $F|_Y = f$. Thus it remains to show that $\|F\| = \|f\| = 1$. It needs to show that $|F(z)| \leq \|z\|$ for all $z \in Z$. Note for $z \in Z$, consider the polar form $F(z) = re^{i\theta}$. Then $F(e^{-i\theta}z) = r \in \mathbb{R}$ and thus $F(e^{-i\theta}z) = H(e^{-i\theta}z)$. This yields that

$$|F(z)| = r = |F(e^{-i\theta}z)| = |H(e^{-i\theta}z)| \leq \|H\|\|e^{-i\theta}z\| \leq \|z\|.$$

The proof is finished. □

Remark 5.2. Before completing the proof of the Hahn-Banach Theorem, Let us first recall one of super important results in mathematics, called *Zorn's Lemma*, a very humble name. Every mathematics student should know it.

Zorn's Lemma: Let $\mathcal{X}$ be a non-empty set with a partially order " $\leq$ ". Assume that every totally order subset $\mathcal{C}$ of $\mathcal{X}$ has an upper bound, i.e. there is an element $\mathfrak{z} \in \mathcal{X}$ such that $c \leq \mathfrak{z}$ for all $c \in \mathcal{C}$. Then $\mathcal{X}$ must contain a maximal element $\mathfrak{m}$, that is, if $\mathfrak{m} \leq x$ for some $x \in \mathcal{X}$, then $\mathfrak{m} = x$.

The following is the typical argument of applying the Zorn's Lemma.

Theorem 5.3. Hahn-Banach Theorem : *Let X be a normed space and let Y be a subspace of X. If $f \in Y^*$, then there exists a linear extension $F \in X^*$ of f such that $\|F\| = \|f\|$.*

Proof. Let $\mathcal{X}$ be the collection of the pairs (Y_1, f_1), where $Y \subseteq Y_1$ is a subspace of X and $f_1 \in Y_1^*$ such that $f_1|_Y = f$ and $\|f_1\|_{Y_1^*} = \|f\|_{Y^*}$. Define a partial order $\leq$ on $\mathcal{X}$ by $(Y_1, f_1) \leq (Y_2, f_2)$ if $Y_1 \subseteq Y_2$ and $f_2|_{Y_1} = f_1$. Then by the Zorn's lemma, there is a maximal element $(\widetilde{Y}, F)$ in $\mathcal{X}$. The maximality of $(\widetilde{Y}, F)$ and Lemma 5.1 will give $\widetilde{Y} = X$. The proof is finished. □

Proposition 5.4. *Let X be a normed space and $x_0 \in X$. Then there is $f \in X^*$ with $\|f\| = 1$ such that $f(x_0) = \|x_0\|$. Consequently, we have*

$$\|x_0\| = \sup\{|g(x)| : g \in B_{X^*}\}.$$

Also, if $x, y \in X$ with $x \neq y$, then there exists $f \in X^*$ such that $f(x) \neq f(y)$.

Proof. Let $Y = \mathbb{K}x_0$. Define $f_0 : Y \to \mathbb{K}$ by $f_0(\alpha x_0) := \alpha\|x_0\|$ for $\alpha \in \mathbb{K}$. Then $f_0 \in Y^*$ with $\|f_0\| = \|x_0\|$. So, the result follows from the Hahn-Banach Theorem at once. □

Remark 5.5. Proposition 5.4 tells us that the dual space X^* of X must be non-zero. Indeed, the dual space X^* is very "*Large*" so that it can separate any pair of distinct points in X.
Furthermore, for any normed space Y and any pair of points $x_1, x_2 \in X$ with $x_1 \neq x_2$, we can find an element $T \in B(X, Y)$ such that $Tx_1 \neq Tx_2$. In fact, fix a non-zero element $y \in Y$. Then by Proposition 5.4, there is $f \in X^*$ such that $f(x_1) \neq f(x_2)$. So, if we define $Tx = f(x)y$, then $T \in B(X, Y)$ as desired.

Proposition 5.6. *With the notation as above, if M is closed subspace and $v \in X \setminus M$, then there is $f \in X^*$ such that $f(M) \equiv 0$ and $f(v) \neq 0$.*

Proof. Since M is a closed subspace of X, we can consider the quotient space X/M. Let $\pi : X \to X/M$ be the natural projection. Notice that $\bar{v} := \pi(v) \neq 0 \in X/M$ because $\bar{v} \in X \setminus M$. Then by Corollary 5.4, there is a non-zero element $\bar{f} \in (X/M)^*$ such that $\bar{f}(\bar{v}) \neq 0$. So, the linear functional $f := \bar{f} \circ \pi \in X^*$ is as desired. □

Proposition 5.7. *Using the notation as above, if X^* is separable, then X is separable.*

Proof. Let $F := \{f_1, f_2....\}$ be a dense subset of X^*. Then there is a sequence (x_n) in X with $\|x_n\| = 1$ and $|f_n(x_n)| \geq 1/2\|f_n\|$ for all n. Now let M be the closed linear span of x_n's. Then M is a separable closed subspace of X. We are going to show that $M = X$.
Suppose not. Proposition 5.6 will give us a non-zero element $f \in X^*$ such that $f(M) \equiv 0$. Since $\{f_1, f_2....\}$ is dense in X^*, we have $B(f, r) \cap F \neq \emptyset$ for all $r > 0$. Therefore, if $B(f, r) \cap F \neq$ is finite for some $r > 0$, then $f = f_m$ for some $f_m \in F$. This implies that $\|f\| = \|f_m\| \leq 2|f_m(x_m)| = 2|f(x_m)| = 0$ and thus, $f = 0$ which contradicts to $f \neq 0$.
So, $B(f, r) \cap F$ is infinite for all $r > 0$. In this case, there is a subsequence (f_{n_k}) such that $\|f_{n_k} - f\| \to 0$. This gives

$$\frac{1}{2}\|f_{n_k}\| \leq |f_{n_k}(x_{n_k})| = |f_{n_k}(x_{n_k}) - f(x_{n_k})| \leq \|f_{n_k} - f\| \to 0$$

because $f(M) \equiv 0$. So $\|f_{n_k}\| \to 0$ and hence $f = 0$. It leads to a contradiction again. Thus, we can conclude that $M = X$ as desired. □

Remark 5.8. The converse of Proposition 5.7 does not hold. For example, consider $X = \ell^1$. Then ℓ^1 is separable but the dual space $(\ell^1)^* = \ell^\infty$ is not.

Proposition 5.9. *Let X and Y be normed spaces. For each element $T \in B(X, Y)$, define a linear operator $T^* : Y^* \to X^*$ by*

$$T^*y^*(x) := y^*(Tx)$$

for $y^* \in Y^*$ and $x \in X$. Then $T^* \in B(Y^*, X^*)$ and $\|T^*\| = \|T\|$. In this case, T^* is called the adjoint operator of T.

Proof. We first claim that $\|T^*\| \leq \|T\|$ and hence, $\|T^*\|$ is bounded.
In fact, for any $y^* \in Y^*$ and $x \in X$, we have $|T^*y^*(x)| = |y^*(Tx)| \leq \|y^*\|\|T\|\|x\|$. So, $\|T^*y^*\| \leq \|T\|\|y^*\|$ for all $y^* \in Y^*$. Thus, $\|T^*\| \leq \|T\|$.
It remains to show $\|T\| \leq \|T^*\|$. Let $x \in B_X$. Then by Proposition 5.4, there is $y^* \in S_{X^*}$ such that $\|Tx\| = |y^*(Tx)| = |T^*y^*(x)| \leq \|T^*y^*\| \leq \|T^*\|$. This implies that $\|T\| \leq \|T^*\|$. □

Example 5.10. Let X and Y be the finite dimensional normed spaces. Let $(e_i)_{i=1}^n$ and $(f_j)_{j=1}^m$ be the bases for X and Y respectively. Let $\theta_X : X \to X^*$ and $\theta_Y : X \to Y^*$ be the identifications as in Example 4.2. Let $e_i^* := \theta_X e_i \in X^*$ and $f_j^* := \theta_Y f_j \in Y^*$. Then $e_i^*(e_l) = \delta_{il}$ and $f_j^*(f_l) = \delta_{jl}$, where, $\delta_{il} = 1$ if $i = l$; otherwise is 0.
Now if $T \in B(X,Y)$ and $(a_{ij})_{m\times n}$ is the representative matrix of T corresponding to the bases $(e_i)_{i=1}^n$ and $(f_j)_{j=1}^m$ respectively, then $a_{kl} = f_k^*(Te_l) = T^*f_k^*(e_l)$. Therefore, if $(a'_{lk})_{n\times m}$ is the representative matrix of T^* corresponding to the bases (f_j^*) and (e_i^*), then $a_{kl} = a'_{lk}$. Hence the transpose $(a_{kl})^t$ is the the representative matrix of T^*.

Proposition 5.11. *Let Y be a closed subspace of a normed space X. Let $i : Y \to X$ be the natural inclusion and $\pi : X \to X/Y$ the natural projection. Then*

- *(i) the adjoint operator $i^{**} : Y^{**} \to X^{**}$ is an isometry.*
- *(ii) the adjoint operator $\pi^* : (X/Y)^* \to X^*$ is an isometry.*

Consequently, Y^{**} and $(X/Y)^*$ can be viewed as the closed subspaces of X^{**} and X^* respectively.

Proof. For Part (i), we first notice that for any $x^* \in X^*$, the image i^*x^* in Y^* is just the restriction of x^* on Y, write $x^*|_Y$. Now let $\phi \in Y^{**}$. Then for any $x^* \in X^*$, we have

$$|i^{**}\phi(x^*)| = |\phi(i^*x^*)| = |\phi(x^*|_Y)| \leq \|\phi\|\|x^*|_Y\|_{Y^*} \leq \|\phi\|\|x^*\|_{X^*}.$$

So, $\|i^{**}\phi\| \leq \|\phi\|$. It remains to show the inverse inequality. Now for each $y^* \in Y^*$, the Hahn-Banach Theorem gives an element $x^* \in X^*$ such that $\|x^*\|_{X^*} = \|y^*\|_{Y^*}$ and $x^*|_Y = y^*$ and hence, $i^*x^* = y^*$. Then we have

$$|\phi(y^*)| = |\phi(x^*|_Y)| = |\phi(i^*x^*)| = |(i^{**} \circ \phi)(x^*)| \leq \|i^{**}\phi\|\|x^*\|_{X^*} = \|i^{**}\phi\|\|y^*\|_{Y^*}$$

for all $y^* \in Y^*$. Therefore, we have $\|i^{**}\phi\| = \|\phi\|$.
For Part (ii), let $\psi \in (X/Y)^*$. Notice that since $\|\pi^*\| = \|\pi\| \leq 1$, we have $\|\pi^*\psi\| \leq \|\psi\|$. On the other hand, for each $\bar{x} := \pi(x) \in X/Y$ with $\|\bar{x}\| < 1$, we can choose an element $m \in Y$ such that $\|x + m\| < 1$. So, we have

$$|\psi(\bar{x})| = |\psi \circ \pi(x)| = |\psi \circ \pi(x + m)\| \leq \|\psi \circ \pi\| = \|\pi^*(\psi)\|.$$

Thus we have $\|\psi\| \leq \|\pi^*(\psi)\|$. The proof is finished. □

Remark 5.12. By using Proposition 5.11, we can give an alternative proof of the Riesz's Lemma 2.4.
With the notation as in Proposition 5.11, if $Y \subsetneq X$, then we have $\|\pi\| = \|\pi^*\| = 1$ because π^* is an isometry by Proposition 5.11(ii). Thus we have $\|\pi\| = \sup\{\|\pi(x)\| : x \in X, \ \|x\| = 1\} = 1$. So, for any $0 < \theta < 1$, we can find element $z \in X$ with $\|z\| = 1$ such that $\theta < \|\pi(z)\| = \inf\{\|z+y\| : y \in Y\}$. The Riesz's Lemma follows.

6. Reflexive Spaces

Proposition 6.1. *For a normed space X, let $Q : X \longrightarrow X^{**}$ be the canonical map, that is, $Qx(x^*) := x^*(x)$ for $x^* \in X^*$ and $x \in X$. Then Q is an isometry.*

Proof. Note that for $x \in X$ and $x^* \in B_{X^*}$, we have $|Q(x)(x^*)| = |x^*(x)| \leq \|x\|$. Then $\|Q(x)\| \leq \|x\|$.
It remains to show that $\|x\| \leq \|Q(x)\|$ for all $x \in X$. In fact, for $x \in X$, there is $x^* \in X^*$ with $\|x^*\| = 1$ such that $\|x\| = |x^*(x)| = |Q(x)(x^*)|$ by Proposition 5.4. Thus we have $\|x\| \leq \|Q(x)\|$. The proof is finished. □

Remark 6.2. Let $T : X \to Y$ be a bounded linear operator and $T^{**} : X^{**} \to Y^{**}$ the second dual operator induced by the adjoint operator of T. With notation as in Proposition 6.1 above, the following diagram always commutes.

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ {\scriptstyle Q_X}\big\downarrow & & \big\downarrow{\scriptstyle Q_Y} \\ X^{**} & \xrightarrow{T^{**}} & Y^{**} \end{array}$$

Definition 6.3. *A normed space X is said to be reflexive if the canonical map $Q : X \longrightarrow X^{**}$ is surjective. (Notice that every reflexive space must be a Banach space.)*

Example 6.4. *We have the following examples.*

(i) : Every finite dimensional normed space X is reflexive.
(ii) : ℓ^p is reflexive for $1 < p < \infty$.
(iii) : c_0 and ℓ^1 are not reflexive.

Proof. For Part (i), if $\dim X < \infty$, then $\dim X = \dim X^{**}$. Hence, the canonical map $Q : X \to X^{**}$ must be surjective.
Part (ii) follows from $(\ell^p)^* = \ell^q$ for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.
For Part (iii), notice that $c_0^{**} = (\ell^1)^* = \ell^{**}$. Since ℓ^∞ is non-separable but c_0 is separable. So, the canonical map Q from c_0 to $c_0^{**} = \ell^\infty$ must not be surjective.
For the case of ℓ^1, we have $(\ell^1)^{**} = (\ell^\infty)^*$. Since ℓ^∞ is non-separable, the dual space $(\ell^\infty)^*$ is non-separable by Proposition 5.7. So, $\ell^1 \neq (\ell^1)^{**}$. □

Proposition 6.5. *Every closed subspace of a reflexive space is reflexive.*

Proof. Let Y be a closed subspace of a reflexive space X. Let $Q_Y : Y \to Y^{**}$ and $Q_X : X \to X^{**}$ be the canonical maps as before. Let $y_0^{**} \in Y^{**}$. We define an element $\phi \in X^{**}$ by $\phi(x^*) := y_0^{**}(x^*|_Y)$ for $x^* \in X^*$. Since X is reflexive, there is $x_0 \in X$ such that $Q_X x_0 = \phi$. Suppose $x_0 \notin Y$. Then by Proposition 5.6, there is $x_0^* \in X^*$ such that $x_0^*(x_0) \neq 0$ but $x_0^*(Y) \equiv 0$. Note that we have $x_0^*(x_0) = Q_X x_0(x_0^*) = \phi(x_0^*) = y_0^{**}(x_0^*|_Y) = 0$. It leads to a contradiction. So, $x_0 \in Y$. The proof is finished if we have $Q_Y(x_0) = y_0^{**}$.
In fact, for each $y^* \in Y^*$, then by the Hahn-Banach Theorem, y^* has a continuous extension x^* in X^*. Then we have

$$Q_Y(x_0)(y^*) = y^*(x_0) = x^*(x_0) = Q_X(x_0)(x^*) = \phi(x^*) = y_0^{**}(x^*|_Y) = y_0^{**}(y^*).$$

□

Example 6.6. *By using Proposition 6.5, we immediately see that the space ℓ^∞ is not reflexive because it contains a non-reflexive closed subspace c_0.*

Proposition 6.7. *Let X be a normed space. Then we have the following assertions.*

(i) X is reflexive if and only if the dual space X^* is reflexive.
(ii) If X is reflexive, then so is every quotient of X.

Proof. For Part (i), suppose that X is reflexive first. Let $\widetilde{z} \in X^{***}$. Then the restriction $z := \widetilde{z}|_X \in X^*$. Then one can directly check that $Qz = z$ on X^{**} since $X^{**} = X$.
For the converse, assume that X^* is reflexive but X is not. So, X is a proper closed subspace of X^{**}. Then by using the Hahn-Banach Theorem, we can find a non-zero element $\phi \in X^{***}$ such that $\phi(X) \equiv 0$. However, since X^{***} is reflexive, we have $\phi \in X^*$ and hence, $\phi = 0$ which leads to a contradiction.
For Part (ii), we assume that X is reflexive. Let M be a closed subspace of X and $\pi : X \to X/M$ the natural projection. Notice that the adjoint operator $\pi^* : (X/M)^* \to X^*$ is an isometry (**Check !**). So, $(X/M)^*$ can be viewed as a closed subspace of X^*. So, by Part (i) and Proposition 6.5, we see that $(X/M)^*$ is reflexive. Then X/M is reflexive by using Part (i) again.
The proof is complete. □

Lemma 6.8. *Let M be a closed subspace of a normed space X. Let $r : X^* \to M^*$ be the restriction map, that is $x^* \in X^* \mapsto x^*|_M \in M^*$. Put $M^\perp := \ker r := \{x^* \in X^* : x^*(M) \equiv 0\}$. Then the canonical linear isomorphism $\widetilde{r} : X^*/M^\perp \to M^*$ induced by r is an isometric isomorphism.*

Proof. We first note that r is surjective by using the Hahn-Banach Theorem. It needs to show that $\widetilde{r}$ is an isometry. Notice that $\widetilde{r}(x^* + M^\perp) = x^*|_M$ for all $x^* \in X^*$. Now for any $x^* \in X^*$, we have $\|x^*+y^*\|_{X^*} \geq \|x^*+y^*\|_{M^*} = \|x^*|_M\|_{M^*}$ for all $y^* \in M^\perp$. So we have $\|\widetilde{r}(x^*+M^\perp)\| = \|x^*|_M\|_{M^*} \leq \|x^* + M^\perp\|$. It remains to show the reverse inequality.
Now for any $x^* \in X^*$, then by the Hahn-Banach Theorem again, there is $z^* \in X^*$ such that $z^*|_M = x^*|_M$ and $\|z^*\| = \|x^*|_M\|_{M^*}$. Then $x^* - z^* \in M^\perp$ and hence, we have $x^* + M^\perp = z^* + M^\perp$. This implies that

$$\|x^* + M^\perp\| = \|z^* + M^\perp\| \leq \|z^*\| = \|x^*|_M\|_{M^*} = \|\widetilde{r}(x^* + M^\perp)\|.$$

The proof is complete. □

Proposition 6.9. (Three space property): *Let M be a closed subspace of a normed space X. If M and the quotient space X/M both are reflexive, then so is X.*

Proof. Let $\pi : X \to X/M$ be the natural projection. Let $\psi \in X^{**}$. We going to show that $\psi \in im(Q_X)$. Since $\pi^{**}(\psi) \in (X/M)^{**}$, there exists $x_0 \in X$ such that $\pi^{**}(\phi) = Q_{X/M}(x_0 + M)$ because X/M is reflexive. So we have

$$\pi^{**}(\psi)(\bar{x}^*) = Q_{X/M}(x_0 + M)(\bar{x}^*)$$

for all $\bar{x}^* \in (X/M)^*$. This implies that

$$\psi(\bar{x}^* \circ \pi) = \psi(\pi^*\bar{x}^*) = \pi^{**}(\psi)(\bar{x}^*) = Q_{X/M}(x_0 + M)(\bar{x}^*) = \bar{x}^*(x_0 + M) = Q_X x_0(\bar{x}^* \circ \pi)$$

for all $\bar{x}^* \in (X/M)^*$. Therefore, we have

$$\psi = Q_X x_0 \quad \text{on} \quad M^\perp.$$

So, we have $\psi - Q_X(x_0) \in X^*/M^\perp$. Let $f : M^* \to X^*/M^\perp$ be the inverse of the isometric isomorphism $\widetilde{r}$ which is defined as in Lemma 6.8. Then the composite $(\psi - Q_X x_0) \circ f : M^* \to X^*/M^\perp \to \mathbb{K}$ lies in M^{**}. Then by the reflexivity of M, there is an element $m_0 \in M$ such that

$$(\psi - Q_X x_0) \circ f = Q_M(m_0) \in M^{**}.$$

On the other hand, notice that for each $x^* \in X^*$, we can find an element $m^* \in M^*$ such that $f(m^*)x^* + M|bot \in X^*/M^\perp$ because f is surjective, moreover, by the construction of $\widetilde{r}$ in Lemma 6.8, we see that $x^*|_M = m^*$. This gives

$$\psi(x^*) - x^*(x_0) = (\psi - Q_X x_0)(m^*) \circ f = Q_M(m_0)(m^*) = m^*(m_0) = x^*(m_0).$$

Thus, we have $\psi(x^*) = x^*(x_0+m_0)$ for all $x^* \in X^*$. From this we have $\psi = Q_X(x_0+m_0) \in im(Q_X)$ as desired. The proof is complete. □

7. Weakly convergent and Weak* convergent

Definition 7.1. *Let X be a normed space. A sequence (x_n) is said to be weakly convergent if there is $x \in X$ such that $f(x_n) \to f(x)$ for all $f \in X^*$. In this case, x is called a weak limit of (x_n).*

Proposition 7.2. *A weak limit of a sequence is unique if it exists. In this case, if (x_n) weakly converges to x, write $x = w\text{-}\lim_n x_n$ or $x_n \xrightarrow{w} x$.*

Proof. The uniqueness follows from the Hahn-Banach Theorem immediately. □

Remark 7.3. *It is clear that if a sequence (x_n) converges to $x \in X$ in norm, then $x_n \xrightarrow{w} x$. However, the weakly convergence of a sequence does not imply the norm convergence.*
For example, consider $X = c_0$ and (e_n). Then $f(e_n) \to 0$ for all $f \in c_0^* = \ell^1$ but (e_n) is not convergent in c_0.

Proposition 7.4. *Suppose that X is finite dimensional. A sequence (x_n) in X is norm convergent if and only if it is weakly convergent.*

Proof. Suppose that (x_n) weakly converges to x. Let $\mathcal{B} := \{e_1, .., e_N\}$ be a base for X and let f_k be the k-th coordinate functional corresponding to the base $\mathcal{B}$, that is $v = \sum_{k=1}^N f_k(v)e_k$ for all $v \in X$. Since $\dim X < \infty$, we have f_k in X^* for all $k = 1, ..., N$. Therefore, we have $\lim_n f_k(x_n) = f_k(x)$ for all $k = 1, ..., N$. So, we have $\|x_n - x\| \to 0$. □

Definition 7.5. *Let X be a normed space. A sequence (f_n) in X^* is said to be weak* convergent if there is $f \in X^*$ such that $\lim_n f_n(x) = f(x)$ for all $x \in X$, that is f_n point-wise converges to f. In this case, f is called the weak* limit of (f_n). Write $f = w^*\text{-}\lim_n f_n$ or $f_n \xrightarrow{w^*} f$.*

Remark 7.6. *In the dual space X^* of a normed space X, we always have the following implications:*

"Norm Convergent" $\Longrightarrow$ "Weakly Convergent" $\Longrightarrow$ "Weak Convergent".*

However, the converse of each implication does not hold.

Example 7.7. *Remark 7.3 has shown that the w-convergence does not imply $\|\cdot\|$-convergence. We now claim that the w^*-convergence also Does Not imply the w-convergence.*
Consider $X = c_0$. Then $c_0^* = \ell^1$ and $c_0^{**} = (\ell^1)^* = \ell^\infty$. Let $e_n^* = (0, ...0, 1, 0...) \in \ell^1 = c_0^*$, where the n-th coordinate is 1. Then $e_n^* \xrightarrow{w^*} 0$ but $e_n^* \nrightarrow 0$ weakly because $e^{**}(e_n^*) \equiv 1$ for all n, where $e^{**} := (1, 1, ...) \in \ell^\infty = c_0^{**}$. Hence the w^*-convergence does not imply the w-convergence.

Proposition 7.8. *Let (f_n) be a sequence in X^*. Suppose that X is reflexive. Then $f_n \xrightarrow{w} f$ if and only if $f_n \xrightarrow{w^*} f$.*
In particular, if $\dim X < \infty$, then the followings are equivalent:

(i) : $f_n \xrightarrow{\|\cdot\|} f$;
(ii) : $f_n \xrightarrow{w} f$;
(iii) : $f_n \xrightarrow{w^*} f$.

Theorem 7.9. (**Banach**) : *Let X be a separable normed space. If (f_n) is a bounded sequence in X^*, then it has a w^*-convergent subsequence.*

Proof. Let $D := \{x_1, x_2, ...\}$ be a countable dense subset of X. Note that since $(f_n)_{n=1}^\infty$ is bounded, $(f_n(x_1))$ is a bounded sequence in $\mathbb{K}$. Then $(f_n(x_1))$ has a convergent subsequence, say $(f_{1,k}(x_1))_{k=1}^\infty$ in $\mathbb{K}$. Let $c_1 := \lim_k f_{1,k}(x_1)$. Now consider the bounded sequence $(f_{1,k}(x_2))$. Then there is convergent subsequence, say $(f_{2,k}(x_2))$, of $(f_{1,k}(x_2))$. Put $c_2 := \lim_k f_{2,k}(x_2)$. Notice that we still have $c_1 = \lim_k f_{2,k}(x_1)$. To repeat the same step, if we define $(m,k) \leq (m',k')$ if $m < m'$; or $m = m'$ with $k \leq k'$, we can find a sequence $(f_{m,k})_{m,k}$ in X^* such that

(i) : $(f_{m+1,k})_{k=1}^\infty$ is a subsequence of $(f_{m,k})_{k=1}^\infty$ for $m = 0, 1, ..,$ where $f_{0,k} := f_k$.
(ii) : $c_i = \lim_k f_{m,k}(x_i)$ exists for all $1 \leq i \leq m$.

Now put $h_k := f_{k,k}$. Then (h_k) is a subsequence of (f_n). Notice that for each i, we have $\lim_k h_k(x_i) = \lim_k f_{i,k}(x_i) = c_i$ by the construction (ii) above. Since $(\|h_k\|)$ is bounded and D is dense in X, we have $h(x) := \lim_k h_k(x)$ exists for all $x \in X$ and $h \in X^*$. That is $h = w^*\text{-}\lim_k h_k$. The proof is finished. □

Remark 7.10. *Theorem 7.9 does not hold if the separability of X is removed.*
For example, consider $X = \ell^\infty$ and δ_n the n-th coordinate functional on ℓ^∞. Then $\delta_n \in (\ell^\infty)^*$ with $\|\delta_n\|_{(\ell^\infty)^*} = 1$ for all n. Suppose that (δ_n) has a w^*-convergent subsequence $(\delta_{n_k})_{k=1}^\infty$. Define $x \in \ell^\infty$ by

$$x(m) = \begin{cases} 0 & \text{if } m \neq n_k; \\ 1 & \text{if } m = n_{2k}; \\ -1 & \text{if } m = n_{2k+1}. \end{cases}$$

Hence we have $|\delta_{n_i}(x) - \delta_{n_{i+1}}(x)| = 2$ for all $i = 1, 2, ...$ It leads to a contradiction. So (δ_n) has no w^*-convergent subsequence.

Corollary 7.11. *Let X be a separable space. In X^* assume that the set of all w^*-convergent sequences coincides with the set of all normed convergent sequences, that is a sequence (f_n) is w^*-convergent if and only if it is norm convergent. Then $\dim X < \infty$.*

Proof. It needs to show that the closed unit ball B_{X^*} in X^* is compact in norm. Let (f_n) be a sequence in B_{X^*}. By using Theorem 7.9, (f_n) has a w^*-convergent subsequence (f_{n_k}). Then by the assumption, (f_{n_k}) is norm convergent. Note that if $\lim_k f_{n_k} = f$ in norm, then $f \in B_{X^*}$. So B_{X^*} is compact and thus $\dim X^* < \infty$. So $\dim X^{**} < \infty$ that gives $\dim X$ is finite because $X \subseteq X^{**}$. □

Corollary 7.12. *Suppose that X is a separable. If X is reflexive space, then the closed unit ball B_X of X is sequentially weakly compact, i.e. it is equivalent to saying that any bounded sequence in X has a weakly convergent subsequence.*

Proof. Let $Q : X \to X^{**}$ be the canonical map as before. Let (x_n) be a bounded sequence in X. Hence, (Qx_n) is a bounded sequence in X^{**}. We first notice that since X is reflexive and separable, X^* is also separable by Proposition 5.7. So, we can apply Theorem 7.9, (Qx_n) has a w^*-convergent subsequence (Qx_{n_K}) in $X^{**} = Q(X)$ and hence, (x_{n_k}) is weakly convergent in X. □

8. Open Mapping Theorem

Let E and F be the metric spaces. Recall that a mapping $f : E \to F$ is called an *open mapping* if $f(U)$ is an open subset of F whenever U is an open subset of E.
It is clear that a continuous bijection is a homeomorphism if and only if it is an open map.

Remark 8.1. Warning *An open map need not be a closed map.*
For example, let $p : (x,y) \in \mathbb{R}^2 \mapsto x \in \mathbb{R}$. Then p is an open map but it is not a closed map. In fact, if we let $A = \{(x, 1/x) : x \neq 0\}$, then A is closed but $p(A) = \mathbb{R} \setminus \{0\}$ is not closed.

Lemma 8.2. *Let X and Y be normed spaces and $T : X \to Y$ a linear map. Then T is open if and only if 0 is an interior point of $T(U)$ where U is the open unit ball of X.*

Proof. The necessary condition is obvious.
For the converse, let W be a non-empty subset of X and $a \in W$. Put $b = Ta$. Since W is open, we choose $r > 0$ such that $B_X(a, r) \subseteq W$. Notice that $U = \frac{1}{r}(B_X(a, r) - a) \subseteq \frac{1}{r}(W - a)$. So, we have $T(U) \subseteq \frac{1}{r}(T(W) - b)$. Then by the assumption, there is $\delta > 0$ such that $B_Y(0, \delta) \subseteq T(U) \subseteq \frac{1}{r}(T(W) - b)$. This implies that $b + rB_Y(0, \delta) \subseteq T(W)$ and so, $T(a) = b$ is an interior point of $T(W)$. □

Corollary 8.3. *Let M be a closed subspace of a normed space X. Then the natural projection $\pi : X \to X/M$ is an open map.*

Proof. Put U and V the open unit balls of X and X/M respectively. Using Lemma 8.2, the result is obtained by showing that $V \subseteq \pi(U)$. Note that if $\bar{x} = \pi(x) \in V$, then by the definition a quotient norm, we can find an element $m \in M$ such that $\|x + m\| < 1$. Hence we have $x + m \in U$ and $\bar{x} = \pi(x + m) \in \pi(U)$. □

Lemma 8.4. *Let $T : X \longrightarrow Y$ be a bounded linear surjection from a Banach space X onto a Banach space Y. Then 0 is an interior point of $\overline{T(U)}$, where U is the open unit ball of X, that is, $U := \{x \in X : \|x\| < 1\}$.*

Proof. Set $U(r) := \{x \in X : \|x\| < r\}$ for $r > 0$ and so, $U = U(1)$.
Claim 1 : 0 is an interior point of $\overline{T(U(1))}$.
Note that since T is surjective, $Y = \bigcup_{n=1}^{\infty} T(U(n))$. Then by the second category theorem, there exists N such that *int* $\overline{T(U(N))} \neq \emptyset$. Let y' be an interior point of $\overline{T(U(N))}$. Then there is $\eta > 0$ such that $B_Y(y', \eta) \subseteq \overline{T(U(N))}$. Since $B_Y(y', \eta) \cap T(U(N)) \neq \emptyset$, we may assume that $y' \in T(U(N))$. Let $x' \in U(N)$ such that $T(x') = y'$. Then we have

$$0 \in B_Y(y', \eta) - y' \subseteq \overline{T(U(N))} - T(x') \subseteq \overline{T(U(2N))} = 2N\overline{T(U(1))}.$$

So we have $0 \in \frac{1}{2N}(B_Y(y', \eta) - y') \subseteq \overline{T(U(1))}$. Hence 0 is an interior point of $\overline{T(U(1))}$. So Claim 1 follows.
Therefore there is $r > 0$ such that $B_Y(0, r) \subseteq \overline{T(U(1))}$. This implies that we have

$$B_Y(0, r/2^k) \subseteq \overline{T(U(1/2^k))} \tag{8.1}$$

for all $k = 0, 1, 2....$
Claim 2 : $D := B_Y(0, r) \subseteq T(U(3))$.
Let $y \in D$. By Eq 8.1, there is $x_1 \in U(1)$ such that $\|y - T(x_1)\| < r/2$. Then by using Eq 8.1 again, there is $x_2 \in U(1/2)$ such that $\|y - T(x_1) - T(x_2)\| < r/2^2$. To repeat the same steps, there exists is a sequence (x_k) such that $x_k \in U(1/2^{k-1})$ and

$$\|y - T(x_1) - T(x_2) - ... - T(x_k)\| < r/2^k$$

for all k. On the other hand, since $\sum_{k=1}^{\infty} \|x_k\| \leq \sum_{k=1}^{\infty} 1/2^{k-1}$ and X is Banach, $x := \sum_{k=1}^{\infty} x_k$ exists in X and $\|x\| \leq 2$. This implies that $y = T(x)$ and $\|x\| < 3$.
Thus we the result follows. □

Theorem 8.5. Open Mapping Theorem : *Retains the notation as in Lemma 8.4. Then T is an open mapping.*

Proof. The proof is finished by using Lemmas 8.2 and 8.4 at once. □

Proposition 8.6. *Let T be a bounded linear isomorphism between Banach spaces X and Y. Then T^{-1} must be bounded.*
Consequently, if $\|\cdot\|$ and $\|\cdot\|'$ both are complete norms on X such that $\|\cdot\| \leq c\|\cdot\|'$ for some $c > 0$, then these two norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent.

Proof. The first assertion follows from the Open Mapping Theorem at once.
Therefore, the last assertion can be obtained by considering the identity map $I : (X, \|\cdot\|) \to (X, \|\cdot\|')$ which is bounded by the assumption. $\square$

Corollary 8.7. *Let X and Y be Banach spaces and $T : X \to Y$ a bounded linear operator. Then the image of T is closed in Y if and only if there is $c > 0$ such that*

$$d(x, \ker T) \leq c\|Tx\|$$

for all $x \in X$.

Proof. Let Z be the image of T. Then the canonical map $\widetilde{T} : X/kerT \to Z$ induced by T is a bounded linear isomorphism. Notice that $\widetilde{T}(\bar{x}) = Tx$ for all $x \in X$, where $\bar{x} := x + \ker T \in X/\ker T$. Now suppose that Z is closed. Then Z becomes a Banach space. Then the Open Mapping Theorem implies that the inverse of $\widetilde{T}$ is also bounded. So, there is $c > 0$ such that $d(x, \ker T) = \|\bar{x}\|_{X/\ker T} \leq c\|\widetilde{T}(\bar{x})\| = c\|T(x)\|$ for all $x \in X$. So, the necessary condition follows.
For the converse, let (x_n) be a sequence in X such that $\lim Tx_n = y \in Y$ exists and so, (Tx_n) is a Cauchy sequence in Y. Then by the assumption, $(\bar{x}_n)$ is a Cauchy sequence in $X/\ker T$. Since $X/\ker T$ is complete, we can find an element $x \in X$ such that $\lim \bar{x}_n = \bar{x}$ in $X/\ker T$. This gives $y = \lim T(x_n) = \lim \widetilde{T}(\bar{x}_n) = \widetilde{T}(\bar{x}) = T(x)$. So, $y \in Z$. The proof is finished. $\square$

9. Closed Graph Theorem

Let $T : X \longrightarrow Y$. The *graph* of T, write $\mathcal{G}(T)$ is defined by the set $\{(x, y) \in X \times Y : y = T(x)\}$. Now the direct sum $X \oplus Y$ is endowed with the norm $\|\cdot\|_\infty$, that is $\|x \oplus y\|_\infty := \max(\|x\|_X, \|y\|_Y)$. We also write $X \oplus_\infty Y$ when $X \oplus Y$ is equipped with this norm.
We say that an operator $T : X \longrightarrow Y$ is said to be closed if its graph $\mathcal{G}(T)$ is a closed subset of $X \oplus_\infty Y$, that is, if a sequence (x_n) of X satisfying the condition $\|(x_n, Tx_n) - (x, y)\|_\infty \to 0$ for some $x \in X$ and $y \in Y$ implies $T(x) = y$.

Theorem 9.1. Closed Graph Theorem : *Let $T : X \longrightarrow Y$ be a linear operator from a Banach space X to a Banach Y. Then T is bounded if and only if T is closed.*

Proof. The part ($\Rightarrow$) is clear.
Assume that T is closed, that is, the graph $\mathcal{G}(T)$ is $\|\cdot\|_\infty$-closed. Define $\|\cdot\|_0 : X \longrightarrow [0, \infty)$ by

$$\|x\|_0 = \|x\| + \|T(x)\|$$

for $x \in X$. Then $\|\cdot\|_0$ is a norm on X. Let $I : (X, \|\cdot\|_0) \longrightarrow (X, \|\cdot\|)$ be the identity operator. It is clear that I is bounded since $\|\cdot\| \leq \|\cdot\|_0$.
Claim: $(X, \|\cdot\|_0)$ is Banach. In fact, let (x_n) be a Cauchy sequence in $(X, \|\cdot\|_0)$. Then (x_n) and $(T(x_n))$ both are Cauchy sequences in $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$. Since X and Y are Banach spaces, there are $x \in X$ and $y \in Y$ such that $\|x_n - x\|_X \to 0$ and $\|T(x_n) - y\|_Y \to 0$. Thus $y = T(x)$ since the graph $\mathcal{G}(T)$ is closed.
Then by Theorem 8.6, the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. So, there is $c > 0$ such that $\|T(\cdot)\| \leq \|\cdot\|_0 \leq c\|\cdot\|$ and hence, T is bounded since $\|T(\cdot)\| \leq \|\cdot\|_0$. The proof is finished. $\square$

Example 9.2. *Let $D := \{\mathbf{c} = (c_n) \in \ell^2 : \sum_{n=1}^\infty n^2|c_n|^2 < \infty\}$. Define $T : D \longrightarrow \ell^2$ by $T(\mathbf{c}) = (nc_n)$. Then T is an unbounded closed operator.*

Proof. Note that since $\|Te_n\| = n$ for all n, T is not bounded. Now we claim that T is closed.
Let $(\mathbf{x_i})$ be a convergent sequence in D such that $(T\mathbf{x_i})$ is also convergent in ℓ^2. Write $\mathbf{x_i} = (x_{i,n})_{n=1}^\infty$ with $\lim\limits_i \mathbf{x_i} = \mathbf{x} := (x_n)$ in D and $\lim\limits_i T\mathbf{x_i} = \mathbf{y} := (y_n)$ in ℓ^2. This implies that if we fix n_0, then

$\lim_i x_{i,n_0} = x_{n_0}$ and $\lim_i n_0 x_{i,n_0} = y_{n_0}$. This gives $n_0 x_{n_0} = y_{n_0}$. Thus $T\mathbf{x} = \mathbf{y}$ and hence T is closed. □

Example 9.3. *Let* $X := \{f \in C^b(0,1) \cap C^\infty(0,1) : \ f' \in C^b(0,1)\}$. *Define* $T : f \in X \mapsto f' \in C^b(0,1)$. *Suppose that* X *and* $C^b(0,1)$ *both are equipped with the* sup*-norm. Then* T *is a closed unbounded operator.*

Proof. Note that if a sequence $f_n \to f$ in X and $f_n' \to g$ in $C^b(0,1)$. Then $f' = g$. Hence T is closed. In fact, if we fix some $0 < c < 1$, then by the Fundamental Theorem of Calculus, we have

$$0 = \lim_n (f_n(x) - f(x)) = \lim_n (\int_c^x (f_n'(t) - f'(t))dt) = \int_c^x (g(t) - f'(t))dt$$

for all $x \in (0,1)$. This implies that we have $\int_c^x g(t)dt = \int_c^x f'(t)dt$. So $g = f'$ on $(0,1)$.
On the other hand, since $\|Tx^n\|_\infty = n$ for all $n \in \mathbb{N}$. Thus T is unbounded as desired. □

10. Uniform Boundedness Theorem

Theorem 10.1. Uniform Boundedness Theorem : *Let* $\{T_i : X \longrightarrow Y : i \in I\}$ *be a family of bounded linear operators from a Banach space* X *into a normed space* Y. *Suppose that for each* $x \in X$, *we have* $\sup_{i\in I} \|T_i(x)\| < \infty$. *Then* $\sup_{i\in I} \|T_i\| < \infty$.

Proof. For each $x \in X$, define

$$\|x\|_0 := \max(\|x\|, \sup_{i\in I} \|T_i(x)\|).$$

Then $\|\cdot\|_0$ is a norm on X and $\|\cdot\| \le \|\cdot\|_0$ on X. If $(X, \|\cdot\|_0)$ is complete, then by the Open Mapping Theorem. This implies that $\|\cdot\|$ is equivalent to $\|\cdot\|_0$ and thus there is $c > 0$ such that

$$\|T_j(x)\| \le \sup_{i\in I} \|T_i(x)\| \le \|x\|_0 \le c\|x\|$$

for all $x \in X$ and for all $j \in I$. So $\|T_j\| \le c$ for all $j \in I$ is as desired.
Thus it remains to show that $(X, \|\cdot\|_0)$ is complete. In fact, if (x_n) is a Cauchy sequence in $(X, \|\cdot\|_0)$, then it is also a Cauchy sequence with respect to the norm $\|\cdot\|$ on X. Write $x := \lim_n x_n$ with respect to the norm $\|\cdot\|$. Also for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\|T_i(x_n - x_m)\| < \varepsilon$ for all $m, n \ge N$ and for all $i \in I$. Now fixing $i \in I$ and $n \ge N$ and taking $m \to \infty$, we have $\|T_i(x_n - x)\| \le \varepsilon$ and thus $\sup_{i\in I} \|T_i(x_n - x)\| \le \varepsilon$ for all $n \ge N$. So we have $\|x_n - x\|_0 \to 0$ and hence $(X, \|\cdot\|_0)$ is complete. The proof is finished. □

Remark 10.2. *Consider* $c_{00} := \{\mathbf{x} = (x_n) : \exists\ N, \forall\ n \ge N; x_n \equiv 0\}$ *which is endowed with* $\|\cdot\|_\infty$. *Now for each* $k \in \mathbb{N}$, *if we define* $T_k \in c_{00}^*$ *by* $T_k((x_n)) := kx_k$, *then* $\sup_k |T_k(\mathbf{x})| < \infty$ *for each* $\mathbf{x} \in c_{00}$ *but* $(\|T_k\|)$ *is not bounded, in fact,* $\|T_k\| = k$. *Thus the assumption of the completeness of* X *in Theorem 10.1 is essential.*

Corollary 10.3. *Let* X *and* Y *be as in Theorem 10.1. Let* $T_k : X \longrightarrow Y$ *be a sequence of bounded operators. Assume that* $\lim_k T_k(x)$ *exists in* Y *for all* $x \in X$. *Then there is* $T \in B(X,Y)$ *such that* $\lim_k \|(T - T_k)x\| = 0$ *for all* $x \in X$. *Moreover, we have* $\|T\| \le \liminf_k \|T_k\|$.

Proof. Notice that by the assumption, we can define a linear operator T from X to Y given by $Tx := \lim_k T_k x$ for $x \in X$. It needs to show that T is bounded. In fact, $(\|T_k\|)$ is bounded by the Uniform Boundedness Theorem since $\lim_k T_k x$ exists for all $x \in X$. So for each $x \in B_X$, there is a positive integer K such that $\|Tx\| \le \|T_K x\| + 1 \le (\sup_k \|T_k\|) + 1$. Thus, T is bounded.
Finally, it remains to show the last assertion. In fact, notice that for any $x \in B_X$ and $\varepsilon > 0$, there is $N(x) \in \mathbb{N}$ such that $\|Tx\| < \|T_k x\| + \varepsilon < \|T_k\| + \varepsilon$ for all $k \ge N(x)$. This gives $\|Tx\| \le \inf_{k\ge N(x)} \|T_k\| + \varepsilon$ for all $k \ge N(x)$ and hence, $\|Tx\| \le \inf_{k\ge N(x)} \|T_k\| + \varepsilon \le \sup_n \inf_{k\ge n} \|T_k\| + \varepsilon$ for all $x \in B_X$ and $\varepsilon > 0$. So, we have $\|T\| \le \liminf_k \|T_k\|$ as desired. □

Corollary 10.4. *Every weakly convergent sequence in a normed space must be bounded.*

Proof. Let (x_n) be a weakly convergent sequence in a normed space X. If we let $Q : X \to X^{**}$ be the canonical isometry, then (Qx_n) is a bounded sequence in X^{**}. Notice that (x_n) is weakly convergent if and only if (Qx_n) is w^*-convergent. So, $(Qx_n(x^*))$ is bounded for all $x^* \in X^*$. Notice that the dual space X^* must be complete. So, we can apply the Uniform Boundedness Theorem to see that (Qx_n) is bounded and so is (x_n). □

11. Geometry of Hilbert space I

From now on, all vectors spaces are over the complex field. Recall that an *inner product* on a vector space V is a function $(\cdot,\cdot) : V \times V \to \mathbb{C}$ which satisfies the following conditions.

(i) $(x, x) \geq 0$ for all $x \in V$ and $(x, x) = 0$ if and only if $x = 0$.
(ii) $\overline{(x, y)} = (y, x)$ for all $x, y \in V$.
(iii) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$ for all $x, y, z \in V$ and $\alpha, \beta \in \mathbb{C}$.

Consequently, for each $x \in V$, the map $y \in V \mapsto (x, y) \in \mathbb{C}$ is conjugate linear by the conditions *(ii)* and *(iii)*, that is $(x, \alpha y + \beta z) = \bar{\alpha}(x, y) + \bar{\beta}(x, z)$ for all $y, z \in V$ and $\alpha, \beta \in \mathbb{C}$.
Also, the inner product $(\cdot,\cdot)$ will give a norm on V which is defined by

$$\|x\| := \sqrt{(x, x)}$$

for $x \in V$.

We first recall the following useful properties of an inner product space which can be found in the standard text books of linear algebras.

Proposition 11.1. *Let V be an inner product space. For all $x, y \in V$, we always have:*

(i): (**Cauchy-Schwarz inequality**)*:* $|(x, y)| \leq \|x\|\|y\|$ *Consequently, the inner product on $V \times V$ is jointly continuous.*

(ii): (**Parallelogram law**)*:* $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$

Furthermore, a norm $\|\cdot\|$ on a vector space X is induced by an inner product if and only if it satisfies the Parallelogram law. In this case such inner product is given by the following:

$$\mathcal{R}e(x, y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2) \quad \textit{and} \quad \mathcal{I}m(x, y) = \frac{1}{4}(\|x + iy\|^2 - \|x - iy\|^2)$$

for all $x, y \in X$.

Example 11.2. *It follows from Proposition 11.1 immediately that ℓ^2 is a Hilbert space and ℓ^p is not for all $p \in [1, \infty] \setminus \{2\}$.*

From now on, all vector spaces are assumed to be a complex inner product spaces. Recall that two vectors x and y in an inner product space V are said to be *orthogonal* if $(x, y) = 0$.

Proposition 11.3. (**Bessel′s inequality**) : *Let $\{e_1, ..., e_N\}$ be an orthonormal set in an inner product space V, that is $(e_i, e_j) = 1$ if $i = j$, otherwise is equal to 0. Then for any $x \in V$, we have*

$$\sum_{i=1}^{N} |(x, e_i)|^2 \leq \|x\|^2.$$

Proof. It can be obtained by the following equality immediately

$$\|x - \sum_{i=1}^{N} (x, e_i)e_i\|^2 = \|x\|^2 - \sum_{i=1}^{N} |(x, e_i)|^2.$$

□

Corollary 11.4. *Let $(e_i)_{i \in I}$ be an orthonormal set in an inner product space V. Then for any element $x \in V$, the set*

$$\{i \in I : (e_i, x) \neq 0\}$$

is countable.

Proof. Note that for each $x \in V$, we have

$$\{i \in I : (e_i, x) \neq 0\} = \bigcup_{n=1}^{\infty} \{i \in I : |(e_i, x)| \geq 1/n\}.$$

Then the Bessel's inequality implies that the set $\{i \in I : |(e_i, x)| \geq 1/n\}$ must be finite for each $n \geq 1$. So the result follows. □

The following is one of the most important classes in mathematics.

Definition 11.5. *A Hilbert space is a Banach space whose norm is given by an inner product.*

In the rest of this section, X always denotes a complex Hilbert space with an inner product $(\cdot, \cdot)$.

Proposition 11.6. *Let (e_n) be a sequence of orthonormal vectors in a Hilbert space X. Then for any $x \in V$, the series $\sum_{n=1}^{\infty}(x, e_n)e_n$ is convergent.*
Moreover, if $(e_{\sigma(n)})$ is a rearrangement of (e_n), that is, $\sigma : \{1, 2...\} \longrightarrow \{1, 2, ..\}$ is a bijection. Then we have

$$\sum_{n=1}^{\infty}(x, e_n)e_n = \sum_{n=1}^{\infty}(x, e_{\sigma(n)})e_{\sigma(n)}.$$

Proof. Since X is a Hilbert space, the convergence of the series $\sum_{n=1}^{\infty}(x, e_n)e_n$ follows from the Bessel's inequality at once. In fact, if we put $s_p := \sum_{n=1}^{p}(x, e_n)e_n$, then we have

$$\|s_{p+k} - s_p\|^2 = \sum_{p+1 \leq n \leq p+k} |(x, e_n)|^2.$$

Now put $y = \sum_{n=1}^{\infty}(x, e_n)e_n$ and $z = \sum_{n=1}^{\infty}(x, e_{\sigma(n)})e_{\sigma(n)}$. Notice that we have

$$\begin{aligned}
(y, y - z) &= \lim_N (\sum_{n=1}^{N}(x, e_n)e_n, \sum_{n=1}^{N}(x, e_n)e_n - z) \\
&= \lim_N \sum_{n=1}^{N} |(x, e_n)|^2 - \lim_N \sum_{n=1}^{N}(x, e_n) \sum_{j=1}^{\infty} \overline{(x, e_{\sigma(j)})}(e_n, e_{\sigma(j)}) \\
&= \sum_{n=1}^{\infty} |(x, e_n)|^2 - \lim_N \sum_{n=1}^{N}(x, e_n)\overline{(x, e_n)} \quad \text{(N.B: for each } n\text{, there is a unique } j \text{ such that } n = \sigma(j)) \\
&= 0.
\end{aligned}$$

Similarly, we have $(z, y - z) = 0$. The result follows. □

A family of an orthonormal vectors, say $\mathcal{B}$, in X is said to be **complete** if it is maximal with respect to the set inclusion order,that is, if $\mathcal{C}$ is another family of orthonormal vectors with $\mathcal{B} \subseteq \mathcal{C}$, then $\mathcal{B} = \mathcal{C}$.
A complete orthonormal subset of X is also called an **orthonormal base** of X.

Proposition 11.7. *Let $\{e_i\}_{i \in I}$ be a family of orthonormal vectors in X. Then the followings are equivalent:*

(i): $\{e_i\}_{i \in I}$ is complete;
(ii): if $(x, e_i) = 0$ for all $i \in I$, then $x = 0$;
(iii): for any $x \in X$, we have $x = \sum_{i \in I}(x, e_i)e_i$;

(iv): for any $x \in X$, we have $\|x\|^2 = \sum_{i\in I} |(x, e_i)|^2$.
In this case, the expression of each element $x \in X$ in Part (iii) is unique.

Note : *there are only countable many $(x, e_i) \neq 0$ by Corollary 11.4, so the sums in (iii) and (iv) are convergent by Proposition 11.6.*

Proposition 11.8. *Let X be a Hilbert space. Then*
(i) : X processes an orthonormal base.
(ii) : If $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ both are the orthonormal bases for X, then I and J have the same cardinality. In this case, the cardinality $|I|$ of I is called the orthonormal dimension of X.

Proof. Part (*i*) follows from Zorn's Lemma at once.
For part (*ii*), if the cardinality $|I|$ is finite, then the assertion is clear since $|I| = \dim X$ (*vector space dimension*) in this case.
Now assume that $|I|$ is infinite, for each e_i, put $J_{e_i} := \{j \in J : (e_i, f_j) \neq 0\}$. Note that since $\{e_i\}_{i\in I}$ is maximal, Proposition 11.7 implies that we have

$$\{f_j\}_{j\in J} \subseteq \bigcup_{i\in I} J_{e_i}.$$

Notice that J_{e_i} is countable for each e_i by using Proposition 11.4. On the other hand, we have $|\mathbb{N}| \leq |I|$ because $|I|$ is infinite and thus $|\mathbb{N} \times I| = |I|$. Then we have

$$|J| \leq \sum_{i\in I} |J_{e_i}| = \sum_{i\in I} |\mathbb{N}| = |\mathbb{N} \times I| = |I|.$$

From symmetry argument, we also have $|I| \leq |J|$. ☐

Remark 11.9. *Recall that a vector space dimension of X is defined by the cardinality of a maximal linearly independent set in X.*
Notice that if X is finite dimensional, then the orthonormal dimension is the same as the vector space dimension.
Also, the vector space dimension is larger than the orthornormal dimension in general since every orthogonal set must be linearly independent.

We say that two Hilbert spaces X and Y are said to be *isomorphic* if there is linear isomorphism U from X onto Y such that $(Ux, Ux') = (x, x')$ for all $x, x' \in X$. In this case U is called a *unitary operator.*

Theorem 11.10. *Two Hilbert spaces are isomorphic if and only if they have the same orthonornmal dimension.*

Proof. The converse part ($\Leftarrow$) is clear.
Now for the ($\Rightarrow$) part, let X and Y be isomorphic Hilbert spaces. Let $U : X \longrightarrow Y$ be a unitary. Note that if $\{e_i\}_{i\in I}$ is an orthonormal base of X, then $\{Ue_i\}_{i\in I}$ is also an orthonormal base of Y. Thus the necessary part follows from Proposition 11.8 at once. ☐

Corollary 11.11. *Every separable Hilbert space is isomorphic to ℓ^2 or $\mathbb{C}^n$ for some n.*

Proof. Let X be a separable Hilbert space.
If $\dim X < \infty$, then it is clear that X is isomorphic to $\mathbb{C}^n$ for $n = \dim X$.
Now suppose that $\dim X = \infty$ and its orthonormal dimension is larger than $|\mathbb{N}|$, that is X has an orthonormal base $\{f_i\}_{i\in I}$ with $|I| > |\mathbb{N}|$. Note that since $\|f_i - f_j\| = \sqrt{2}$ for all $i, j \in I$ with $i \neq j$. This implies that $B(e_i, 1/4) \cap B(e_j, 1/4) = \emptyset$ for $i \neq j$.

On the other hand, if we let D be a countable dense subset of X, then $B(f_i, 1/4) \cap D \neq \emptyset$ for all $i \in I$. So for each $i \in I$, we can pick up an element $x_i \in D \cap B(f_i, 1/4)$. Therefore, one can define an injection from I into D. It is absurd to the countability of D. □

12. Geometry of Hilbert space II

In this section, let X always denote a complex Hilbert space.

Proposition 12.1. *If D is a closed convex subset of X, then there is a unique element $z \in D$ such that*

$$\|z\| = \inf\{\|x\| : x \in D\}.$$

Consequently, for any element $u \in X$, there is a unique element $w \in D$ such that

$$\|u - w\| = d(u, D) := \inf\{\|u - x\| : x \in D\}.$$

Proof. We first claim the existence of such z.
Let $d := \inf\{\|x\| : x \in D\}$. Then there is a sequence (x_n) in D such that $\|x_n\| \to d$. Notice that (x_n) is a Cauchy sequence. In fact, the Parallelogram Law implies that

$$\|\frac{x_m - x_n}{2}\|^2 = \frac{1}{2}\|x_m\|^2 + \frac{1}{2}\|x_n\|^2 - \|\frac{x_m + x_n}{2}\|^2 \leq \frac{1}{2}\|x_m\|^2 + \frac{1}{2}\|x_n\|^2 - d^2 \longrightarrow 0$$

as $m, n \to \infty$, where the last inequality holds because D is convex and hence $\frac{1}{2}(x_m + x_n) \in D$. Let $z := \lim_n x_n$. Then $\|z\| = d$ and $z \in D$ because D is closed.
For the uniqueness, let $z, z' \in D$ such that $\|z\| = \|z'\| = d$. Thanks to the Parallelogram Law again, we have

$$\|\frac{z - z'}{2}\|^2 = \frac{1}{2}\|z\|^2 + \frac{1}{2}\|z'\|^2 - \|\frac{z + z'}{2}\|^2 \leq \frac{1}{2}\|z\|^2 + \frac{1}{2}\|z'\|^2 - d^2 = 0.$$

Therefore $z = z'$.
The last assertion follows by considering the closed convex set $u - D := \{u - x : x \in D\}$ immediately. □

Proposition 12.2. *Suppose that M is a closed subspace. Let $u \in X$ and $w \in M$. Then the followings are equivalent:*

(i): $\|u - w\| = d(u, M)$;
(ii): $u - w \perp M$, that is $(u - w, x) = 0$ for all $x \in M$.

Consequently, for each element $u \in X$, there is a unique element $w \in M$ such that $u - w \perp M$.

Proof. Let $d := d(u, M)$.
For proving $(i) \Rightarrow (ii)$, fix an element $x \in M$. Then for any $t > 0$, note that since $w + tx \in M$, we have

$$d^2 \leq \|u - w - tx\|^2 = \|u - w\|^2 + \|tx\|^2 - 2Re(u - w, tx) = d^2 + \|tx\|^2 - 2Re(u - w, tx).$$

This implies that

$$2Re(u - w, x) \leq t\|x\|^2 \tag{12.1}$$

for all $t > 0$ and for all $x \in M$. So by considering $-x$ in Eq.12.1, we obtain

$$2|Re(u - w, x)| \leq t\|x\|^2.$$

for all $t > 0$. This implies that $Re(u - w, x) = 0$ for all $x \in M$. Similarly, putting $\pm ix$ into Eq.12.1, we have $Im(u - w, x) = 0$. So (ii) follows.
For $(ii) \Rightarrow (i)$, we need to show that $\|u - w\|^2 \leq \|u - x\|^2$ for all $x \in M$. Note that since $u - w \perp M$ and $w \in M$, we have $u - w \perp w - x$ for all $x \in M$. This gives

$$\|u - x\|^2 = \|(u - w) + (w - x)\|^2 = \|u - w\|^2 + \|w - x\|^2 \geq \|u - w\|^2.$$

Part (i) follows.
The last statement is obtained by Proposition 12.1 immediately. □

Theorem 12.3. *Let M be a closed subspace. Put*

$$M^\perp := \{x \in X : x \perp M\}.$$

Then $M^\perp$ is a closed subspace and we have $X = M \oplus M^\perp$.
In this case, $M^\perp$ is called the orthogonal complement of M.

Proof. It is clear that $M^\perp$ is a closed subspace and $M \cap M^\perp = (0)$. It remains to show $X = M + M^\perp$. Let $u \in X$. Then by Proposition 12.2, we can find an element $w \in M$ such that $u - w \perp M$. Thus $u - w \in M^\perp$ and $u = w + (u - w)$. The proof is finished. □

Corollary 12.4. *With the notation as above, an element $x_0 \notin M$ if and only if there is an element $m \in M$ such that $x_0 - m \perp M$.*

Proof. It is clear from Theorem 12.3. □

Corollary 12.5. *If M is a closed subspace of X, then $M^{\perp\perp} = M$.*

Proof. It is clear that $M \subseteq M^{\perp\perp}$ by the definition of $M^{\perp\perp}$. Now if there is $x \in M^{\perp\perp} \setminus M$, then by the decomposition $X = M \oplus M^\perp$ obtained in Theorem12.3, we have $x = y + z$ for some $y \in M$ and $z \in M^\perp$. This implies that $z = x - y \in M^\perp \cap M^{\perp\perp} = (0)$. This gives $x = y \in M$. It leads to a contradiction. □

Remark 12.6. *It is worthwhile pointing out that for a general Banach space X and a closed subspace M of X, it* **May Not** *have a complementary* **Closed** *subspace N of M, that is $X = M \oplus N$. If M has a complementary closed subspace X, we say that M is complemented in X.*

Example 12.7. *If M is a finite dimensional subspace of a normed space X, then M is complemented in X.*
In fact, if M is spanned by $\{e_i : i = 1, 2.., m\}$, then M is closed and by the Hahn-Banach Theorem, for each $i = 1, ..., m$, there is $e_i^* \in X^*$ such that $e_i^*(e_j) = 1$ if $i = j$, otherwise, it is equal to 0. Put $N := \bigcap_{i=1}^m \ker e_i^*$. Then $X = M \oplus N$.

Example 12.8. (**Very Not Obvious** !!!) *c_0 is not complemented in ℓ^∞.*

Theorem 12.9. Riesz Representation Theorem : *For each $f \in X^*$, then there is a unique element $v_f \in X$ such that*

$$f(x) = (x, v_f)$$

for all $x \in X$ and we have $\|f\| = \|v_f\|$.
Furthermore, if $(e_i)_{i \in I}$ is an orthonormal base of X, then $v_f = \sum_i \overline{f(e_i)} e_i$.

Proof. We first prove the uniqueness of v_f. If $z \in X$ also satisfies the condition: $f(x) = (x, z)$ for all $x \in X$. This implies that $(x, z - v_f) = 0$ for all $x \in X$. So $z - v_f = 0$.
Now for proving the existence of v_f, it suffices to show the case $\|f\| = 1$. Then $\ker f$ is a closed proper subspace. Then by the orthogonal decomposition again, we have

$$X = \ker f \oplus (\ker f)^\perp.$$

Since $f \neq 0$, we have $(\ker f)^\perp$ is linear isomorphic to $\mathbb{C}$. Also note that the restriction of f on $(\ker f)^\perp$ is of norm one. Hence there is an element $v_f \in (\ker f)^\perp$ with $\|v_f\| = 1$ such that $f(v_f) = \|f|_{(\ker f)^\perp}\| = 1$ and $(\ker f)^\perp = \mathbb{C} v_f$. So for each element $x \in X$, we have $x = z + \alpha v_f$ for

some $z \in \ker f$ and $\alpha \in \mathbb{C}$. Then $f(x) = \alpha f(v_f) = \alpha = (x, v_f)$ for all $x \in X$.
Concerning about the last assertion, if we put $v_f = \sum_{i \in I} \alpha_i e_i$, then $f(e_j) = (e_j, v_f) = \overline{\alpha_j}$ for all $j \in I$. The proof is finished. □

Corollary 12.10. *With the notation as in Theorem 12.9, Define the map*

$$\Phi : f \in X^* \mapsto v_f \in X, \text{ i.e., } f(y) = (x, \Phi(f)) \tag{12.2}$$

for all $y \in X$ and $f \in X^*$.
And if we define $(f, g)_{X^*} := (v_g, v_f)_X$ for $f, g \in X^*$. Then $(X^*, (\cdot, \cdot)_{X^*})$ becomes a Hilbert space. Moreover, Φ is an anti-unitary operator from X^* onto X, that is Φ satisfies the conditions:

$$\Phi(\alpha f + \beta g) = \overline{\alpha}\Phi(f) + \overline{\beta}\Phi(g) \quad \text{and} \quad (\Phi f, \Phi g)_X = (g, f)_{X^*}$$

for all $f, g \in X^*$ and $\alpha, \beta \in \mathbb{C}$.
Furthermore, if we define $J : x \in X \mapsto f_x \in X^*$, where $f_x(y) := (y, x)$, then J is the inverse of Φ, and hence, J is an isometric conjugate linear isomorphism.

Proof. The result follows immediately from the observation that $v_{f+g} = v_f + v_g$ and $v_{\alpha f} = \overline{\alpha} v_f$ for all $f \in X^*$ and $\alpha \in \mathbb{C}$.
The last assertion is clearly obtained by the Eq.12.2 above. □

Corollary 12.11. *Every Hilbert space is reflexive.*

Proof. Using the notation as in the Riesz Representation Theorem 12.9, let X be a Hilbert space. and $Q : X \to X^{**}$ the canonical isometry. Let $\psi \in X^{**}$. To apply the Riesz Theorem on the dual space X^*, there exists an element $x_0^* \in X^*$ such that

$$\psi(f) = (f, x_0^*)_{X^*}$$

for all $f \in X^*$. By using Corollary 12.10, there is an element $x_0 \in X$ such that $x_0 = v_{x_0^*}$ and thus, we have

$$\psi(f) = (f, x_0^*)_{X^*} = (x_0, v_f)_X = f(x_0)$$

for all $f \in X^*$. Therefore, $\psi = Q(x_0)$ and so, X is reflexive.
The proof is finished. □

Theorem 12.12. *Every bounded sequence in a Hilbert space has a weakly convergent subsequence.*

Proof. Let (x_n) be a bounded sequence in a Hilbert space X and M be the closed subspace of X spanned by $\{x_m : m = 1, 2...\}$. Then M is a separable Hilbert space.
Method I : Define a map by $j_M : x \in M \mapsto j_M(x) := (\cdot, x) \in M^*$. Then $(j_M(x_n))$ is a bounded sequence in M^*. By Banach's result, Proposition 7.9, $(j_M(x_n))$ has a w^*-convergent subsequence $(j_M(x_{n_k}))$. Put $j_M(x_{n_k}) \xrightarrow{w^*} f \in M^*$, that is $j_M(x_{n_k})(z) \to f(z)$ for all $z \in M$. The Riesz Representation will assure that there is a unique element $m \in M$ such that $j_M(m) = f$. So we have $(z, x_{n_k}) \to (z, m)$ for all $z \in M$. In particular, if we consider the orthogonal decomposition $X = M \oplus M^\perp$, then $(x, x_{n_k}) \to (x, m)$ for all $x \in X$ and thus $(x_{n_k}, x) \to (m, x)$ for all $x \in X$. Then $x_{n_k} \to m$ weakly in X by using the Riesz Representation Theorem again.
Method II : We first note that since M is a separable Hilbert space, the second dual M^{**} is also separable by the reflexivity of M. So the dual space M^* is also separable (see Proposition5.7). Let $Q : M \longrightarrow M^{**}$ be the natural canonical mapping. To apply the Banach's result Proposition 7.9 for X^*, then $Q(x_n)$ has a w^*-convergent subsequence, says $Q(x_{n_k})$. This gives an element $m \in M$ such that $Q(m) = w^*\text{-}\lim_k Q(x_{n_k})$ because M is reflexive. So we have $f(x_{n_k}) = Q(x_{n_k})(f) \to Q(m)(f) = f(m)$ for all $f \in M^*$. Using the same argument as in **Method I** again, x_{n_k} weakly converges to m as desired. □

Remark 12.13. *It is well known that we have the following Theorem due to R. C. James (the proof is highly non-trivial):*

A normed space X is reflexive if and only if every bounded sequence in X has a weakly convergent subsequence.

Theorem 12.12 can be obtained by the James's Theorem directly. However, Theorem 12.12 gives a simple proof in the Hilbert spaces case.

13. Operators on a Hilbert space

Throughout this section, all spaces are complex Hilbert spaces. Let $B(X,Y)$ denote the space of all bounded linear operators from X into Y. If $X = Y$, write $B(X)$ for $B(X,X)$.
Let $T \in B(X,Y)$. We will make use the following simple observation:

$$(Tx, y) = 0 \text{ for all } x \in X; y \in Y \quad \text{if and only if} \quad T = 0. \tag{13.1}$$

Therefore, the elements in $B(X,Y)$ are uniquely determined by the Eq.13.1, that is, $T = S$ in $B(X,Y)$ if and only if $(Tx,y) = (Sx,y)$ for all $x \in X$ and $y \in Y$.

Remark 13.1. *For Hilbert spaces H_1 and H_2, we consider their direct sum $H := H_1 \oplus H_2$. If we define the inner product on H by*

$$(x_1 \oplus x_2, y_1 \oplus y_2) := (x_1, y_1)_{H_1} + (x_2, y_2)_{H_2}$$

for $x_1 \oplus x_2$ and $y_1 \oplus y_2$ in H, then H becomes a Hilbert space. Now for each $T \in B(H_1, H_2)$, we can define an element $\tilde{T} \in B(H)$ by $\tilde{T}(x_1 \oplus x_2) := 0 \oplus Tx_1$. So, the space $B(H_1, H_2)$ can be viewed as a closed subspace of $B(H)$. Thus, we can consider the case of $H_1 = H_2$ for studying the space $B(H_1, H_2)$.

Proposition 13.2. *Let $T \in B(X)$. Then we have*

- *(i): $T = 0$ if and only if $(Tx,x) = 0$ for all $x \in X$. Consequently, for $T, S \in B(X)$, $T = S$ if and only if $(Tx,x) = (Sx,x)$ for all $x \in X$.*
- *(ii): $\|T\| = \sup\{|(Tx,y)| : x, y \in X$ with $\|x\| = \|y\| = 1\}$.*

Proof. It is clear that the necessary part in Part (i). Now we are going to the sufficient part in Part (i), that is we assume that $(Tx,x) = 0$ for all $x \in X$. This implies that we have

$$0 = (T(x+iy), x+iy) = (Tx,x) + i(Ty,x) - i(Tx,y) + (Tiy,iy) = i(Ty,x) - i(Tx,y).$$

So we have $(Ty,x) - (Tx,y) = 0$ for all $x, y \in X$. In particular, if we replace y by iy in the equation, then we get $i(Ty,x) - \bar{i}(Tx,y) = 0$ and hence we have $(Ty,x) + (Tx,y) = 0$. Therefore we have $(Tx,y) = 0$.
For part (ii) : Let $\alpha = \sup\{|(Tx,y)| : x, y \in X \text{ with } \|x\| = \|y\| = 1\}$. It is clear that we have $\|T\| \geq \alpha$. It needs to show $\|T\| \leq \alpha$.
In fact, for each $x \in X$ with $\|x\| = 1$, then by the Hahn-Banach Theorem, there is $f \in X^*$ with $\|f\| = 1$ such that $f(Tx) = \|Tx\|$. The Riesz Representation Theorem, we can find an element $y_f \in X$ with $\|y_f\| = \|f\| = 1$ so that we have $\|Tx\| = f(Tx) = (x, y_f) \leq \alpha$ for all $x \in X$ with $\|x\| = 1$. This implies that $\|T\| \leq \alpha$. The proof is finished. □

Proposition 13.3. *Let $T \in B(X)$. Then there is a unique element T^* in $B(X)$ such that*

$$(Tx, y) = (x, T^*y) \tag{13.2}$$

In this case, T^* is called the adjoint operator of T.

Proof. We first show the uniqueness. Suppose that there are S_1, S_2 in $B(X)$ which satisfy the Eq.13.2. Then $(x, S_1y) = (x, S_2y)$ for all $x, y \in X$. Eq.13.1 implies that $S_1 = S_2$.
Finally, we prove the existence. Note that if we fix an element $y \in X$, define the map $f_y(x) := (Tx, y)$ for all $x \in X$. Then $f_y \in X^*$. The Riesz Representation Theorem implies that there is a unique element $y^* \in X$ such that $(Tx, y) = (x, y^*)$ for all $x \in X$ and $\|f_y\| = \|y^*\|$. On the other hand, we have

$$|f_y(x)| = |(Tx, y)| \leq \|T\|\|x\|\|y\|$$

for all $x, y \in X$ and thus $\|f_y\| \leq \|T\|\|y\|$. If we put $T^*(y) := y^*$, then T^* satisfies the Eq.13.2. Also, we have $\|T^*y\| = \|y^*\| = \|f_y\| \leq \|T\|\|y\|$ for all $y \in X$. So $T^* \in B(X)$ with $\|T^*\| \leq \|T\|$ indeed. Hence T^* is as desired. □

Remark 13.4. Let $S, T : X \to X$ be linear operators (without assuming to be bounded). If they satisfy the Eq.13.2 above, i.e.,

$$(Tx, y) = (x, Sy)$$

for all $x, y \in X$. Using the Closed Graph Theorem, one can show that S and T both are automatically bounded.
In fact, let (x_n) be a sequence in X such that $\lim x_n = x$ and $\lim Sx_n = y$ for some $x, y \in X$. Now for any $z \in X$, we have

$$(z, Sx) = (Tz, x) = \lim(Tz, x_n) = \lim(z, Sx_n) = (z, y).$$

Thus $Sx = y$ and hence S is bounded by the Closed Graph Theorem.
Similarly, we can also see that T is bounded.

Remark 13.5. Let $T \in B(X)$. Let $T^t : X^* \to X^*$ be the transpose of T which is defined by $T^t(f) := f \circ T \in X^*$ for $f \in X^*$ (see Proposition 5.9). Then we have the following commutative diagram (**Check!**)

$$\begin{array}{ccc} X & \xrightarrow{T^*} & X \\ {\scriptstyle J_X}\big\downarrow & & \big\downarrow{\scriptstyle J_X} \\ X^* & \xrightarrow{T^t} & X^* \end{array}$$

where $J_X : X \to X^*$ is the anti-unitary given by the Riesz Representation Theorem (see Corollary 12.10).

Proposition 13.6. *Let $T, S \in B(X)$. Then we have*

(i): $T^* \in B(X)$ and $\|T^*\| = \|T\|$.

(ii): The map $T \in B(X) \mapsto T^* \in B(X)$ is an isometric conjugate anti-isomorphism, that is,

$$(\alpha T + \beta S)^* = \overline{\alpha}T^* + \overline{\beta}S^* \quad \textit{for all} \quad \alpha, \beta \in \mathbb{C}; \quad \textit{and} \quad (TS)^* = S^*T^*.$$

(iii): $\|T^*T\| = \|T\|^2$.

Proof. For Part (i), in the proof of Proposition 13.3, we have shown that $\|T^*\| \leq \|T\|$. And the reverse inequality clearly follows from $T^{**} = T$.
The Part (ii) follows from the adjoint operators are uniquely determined by the Eq.13.2 above.
For Part (iii), we always have $\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$. For the reverse inequality, let $x \in B_X$. Then

$$\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) \leq \|T^*Tx\|\|x\| \leq \|T^*T\|.$$

Therefore, we have $\|T\|^2 \leq \|T^*T\|$. □

Example 13.7. *If $X = \mathbb{C}^n$ and $D = (a_{ij})_{n\times n}$ an $n \times n$ matrix, then $D^* = (\overline{a_{ji}})_{n\times n}$. In fact, notice that*

$$a_{ji} = (De_i, e_j) = (e_i, D^*e_j) = \overline{(D^*e_j, e_i)}.$$

So if we put $D^* = (d_{ij})_{n\times n}$, then $d_{ij} = (D^*e_j, e_i) = \overline{a_{ji}}$.

Example 13.8. *Let $\ell^2(\mathbb{N}) := \{x : \mathbb{N} \to \mathbb{C} : \sum_{i=0}^{\infty} |x(i)|^2 < \infty\}$. And put $(x, y) := \sum_{i=0}^{\infty} x(i)\overline{y(i)}$. Define the operator $D \in B(\ell^2(\mathbb{N}))$ (called the unilateral shift) by*

$$Dx(i) = x(i-1)$$

for $i \in \mathbb{N}$ and where we set $x(-1) := 0$, that is $D(x(0), x(1), ...) = (0, x(0), x(1), ...)$.
Then D is an isometry and the adjoint operator D^* is given by

$$D^*x(i) := x(i+1)$$

for $i = 0, 1, ..,$ that is $D^*(x(0), x(1), ...) = (x(1), x(2), ...)$.
Indeed one can directly check that

$$(Dx, y) = \sum_{i=0}^{\infty} x(i-1)\overline{y(i)} = \sum_{j=0}^{\infty} x(j)\overline{y(j+1)} = (x, D^*y).$$

Note that D^* is NOT an isometry.

Example 13.9. *Let $\ell^\infty(\mathbb{N}) = \{x : \mathbb{N} \to \mathbb{C} : \sup_{i\geq 0} |x(i)| < \infty\}$ and $\|x\|_\infty := \sup_{i\geq 0} |x(i)|$. For each $x \in \ell^\infty$, define $M_x \in B(\ell^2(\mathbb{N}))$ by*

$$M_x(\xi) := x \cdot \xi$$

for $\xi \in \ell^2(\mathbb{N})$, where $(x \cdot \xi)(i) := x(i)\xi(i)$; $i \in \mathbb{N}$.
Then $\|M_x\| = \|x\|_\infty$ and $M_x^* = M_{\overline{x}}$, where $\overline{x}(i) := \overline{x(i)}$.

Definition 13.10. *Let $T \in B(X)$ and let I be the identity operator on X. T is said to be*
(i) : selfadjoint if $T^* = T$;
(ii) : normal if $T^*T = TT^*$;
(iii) : unitary if $T^*T = TT^* = I$.

Proposition 13.11. *We have*
(i) : Let $T : X \longrightarrow X$ be a linear operator. T is selfadjoint if and only if

$$(Tx, y) = (x, Ty) \quad \text{for all } x, y \in X. \tag{13.3}$$

(ii) : T is normal if and only if $\|Tx\| = \|T^*x\|$ for all $x \in X$.

Proof. The necessary part of Part (i) is clear.
Now suppose that the Eq.13.3 holds, it needs to show that T is bounded. Indeed, it follows from Remark13.4 at once.
For Part (ii), note that by Proposition 13.2, T is normal if and only if $(T^*Tx, x) = (TT^*x, x)$. So, Part (ii) follows from that

$$\|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) = (TT^*x, x) = (T^*x, T^*x) = \|T^*x\|^2$$

for all $x \in X$. □

Proposition 13.12. *Let $T \in B(H)$. We have the following assertions.*
(i) : T is selfadjoint if and only if $(Tx, x) \in \mathbb{R}$ for all $x \in H$.

(ii) : If T is selfadjoint, then $\|T\| = \sup\{|(Tx,x)| : x \in H \text{ with } \|x\| = 1\}$.

Proof. Part (i) is clearly follows from Proposition13.2.
For Part (ii), if we let $a = \sup\{|(Tx,x)| : x \in H \text{ with } \|x\| = 1\}$, then it is clear that $a \le \|T\|$. We are now going to show the reverse inequality. Since T is selfadjoint, one can directly check that

$$(T(x+y), x+y) - (T(x-y), x-y) = 4Re(Tx,y)$$

for all $x, y \in H$. Thus if $x, y \in H$ with $\|x\| = \|y\| = 1$ and $(Tx,y) \in \mathbb{R}$, then by using the Parallelogram Law, we have

$$|(Tx,y)| \le \frac{a}{4}(\|x+y\|^2 + \|x-y\|^2) = \frac{a}{2}(\|x\|^2 + \|y\|^2) = a. \tag{13.4}$$

Now for $x, y \in H$ with $\|x\| = \|y\| = 1$, by considering the polar form of $(Tx,y) = re^{i\theta}$, the Eq.13.4 gives

$$|(Tx,y)| = |(Tx, e^{i\theta}y)| \le a.$$

Since $\|T\| = \sup\limits_{\|x\|=\|y\|=1} |(Tx,y)|$, we have $\|T\| \le a$ as desired. The proof is finished. ☐

Proposition 13.13. *Let $T \in B(X)$. Then we have*

$$\ker T = (imT^*)^\perp \quad \text{and} \quad (\ker T)^\perp = \overline{imT^*}$$

where imT denotes the image of T.

Proof. The first equality is clearly follows from $x \in \ker T$ if and only if $0 = (Tx,z) = (x, T^*z)$ for all $z \in X$.
On the other hand, it is clear that we have $M^\perp = \overline{M}^\perp$ for any subspace M of X. This together with the first equality and Corollary12.5 will yield the second equality at once. ☐

Proposition 13.14. *Let $(E, \|\cdot\|)$ be a Banach space. Let M and N be the closed subspaces of E such that*

$$E = M \oplus N \qquad \ldots\ldots\ldots\ldots (*)$$

Define an operator $Q : E \longrightarrow E$ by $Q(y+z) = y$ for $y \in M$ and $z \in N$. Then Q is bounded. In this case, Q is called the projection with respect to the decomposition $(*)$.
Furthermore, if E is a Hilbert space, then $N = M^\perp$ (and hence $(*)$ is the orthogonal decomposition of E with respect to M) if and only if Q satisfies the conditions: $Q^2 = Q$ and $Q^* = Q$. And Q is called the orthogonal projection (or projection for simply) with respect to M.

Proof. For showing the boundedness of Q, by using the Closed Graph Theorem, we need to show that if (x_n) is a sequence in E such that $\lim x_n = x$ and $\lim Qx_n = u$ for some $x, u \in E$, then $Qx = u$.
Indeed, if we let $x_n = y_n \oplus z_n$ and $u = v \oplus w$, where $y_n, v \in M$ and $z_n, w \in N$, then $Qx_n = y_n$. Notice that (z_n) is a convergent sequence in E because $z_n = x_n - y_n$ and (x_n) and (y_n) both are convergent. Let $w = \lim z_n$. This implies that

$$x = \lim x_n = \lim(y_n \oplus z_n) = u \oplus w.$$

Since M and N are closed, we have $u \in M$ and $z \in N$. Therefore, we have $Qx = u$ as desired.

For the last assertion, we further assume that E is a Hilbert space.
It is clear from the definition of Q that $Q(y) = y$ and $Q(z) = 0$ for all $y \in M$ and $z \in N$. Thus we have $Q^2 = Q$.
Now if $N = M^\perp$, then for $y, y' \in M$ and $z, z' \in N$, we have

$$(Q(y+z), y'+z') = (y, y') = (y+z, Q(y'+z')).$$

So $Q^* = Q$.
The converse of the last statement follows from Proposition 13.13 at once because $\ker Q = N$ and $imQ = M$.
The proof is complete. □

Proposition 13.15. *When X is a Hilbert space, we put $\mathcal{M}$ the set of all closed subspaces of X and $\mathcal{P}$ the set of all orthogonal projections on X. Now for each $M \in \mathcal{M}$, let P_M be the corresponding projection with respect to the orthogonal decomposition $X = M \oplus M^{\perp}$. Then there is an one-one correspondence between $\mathcal{M}$ and $\mathcal{P}$ which is defined by*

$$M \in \mathcal{M} \mapsto P_M \in \mathcal{P}.$$

Furthermore, if $M, N \in \mathcal{M}$, then we have
(i) : $M \subseteq N$ if and only if $P_M P_N = P_N P_M = P_M$.
(ii) : $M \perp N$ if and only if $P_M P_N = P_N P_M = 0$.

Proof. It first follows from Proposition 13.14 that $P_M \in \mathcal{P}$.
Indeed the inverse of the correspondence is given by the following. If we let $Q \in \mathcal{P}$ and $M = Q(X)$, then M is closed because $M = \ker(I - Q)$ and $I - Q$ is bounded. Also it is clear that $X = Q(X) \oplus (I - Q)X$ with $\ker Q = M^{\perp}$. Hence M is the corresponding closed subspace of X, that is $M \in \mathcal{M}$ and $P_M = Q$ as desired.
For the final assertion, Part (i) and (ii) follow immediately from the orthogonal decompositions $X = M \oplus M^{\perp} = N \oplus N^{\perp}$ and together with the clear facts that $M \subseteq N$ if and only if $N^{\perp} \subseteq M^{\perp}$. □

14. Spectral Theory I

Definition 14.1. *Let E be a normed space and let $T \in B(E)$. The spectrum of T, write $\sigma(T)$, is defined by*

$$\sigma(T) := \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not invertible in } \ B(E)\}.$$

Remark 14.2. *More precise, for a normed space E, an operator $T \in B(E)$ is said to be invertible in $B(E)$ if T is an linear isomorphism and the inverse T^{-1} is also bounded. However, if E is complete, the Open Mapping Theorem assures that the inverse T^{-1} is bounded automatically. So if E is a Banach space and $T \in B(E)$, then $\lambda \notin \sigma(T)$ if and only if $T - \lambda := T - \lambda I$ is an linear isomorphism. So λ lies in the spectrum $\sigma(T)$ if and only if $T - \lambda$ is either not one-one or not surjective.*
In particular, if there is a non-zero element $v \in X$ such that $Tv = \lambda v$, then $\lambda \in \sigma(T)$ and λ is called an eigenvalue of T with eigenvector v.
We also write $\sigma_p(T)$ for the set of all eigenvalue of T and call $\sigma_p(T)$ the point spectrum.

Example 14.3. *Let $E = \mathbb{C}^n$ and $T = (a_{ij})_{n \times n} \in M_n(\mathbb{C})$. Then $\lambda \in \sigma(T)$ if and only if λ is an eigenvalue of T and thus $\sigma(T) = \sigma_p(T)$.*

Example 14.4. *Let $E = (c_{00}(\mathbb{N}), \|\cdot\|_\infty)$ (note that $c_{00}(\mathbb{N})$ is not a Banach space). Define the map $T : c_{00}(\mathbb{N}) \to c_{00}(\mathbb{N})$ by*

$$Tx(k) := \frac{x(k)}{k+1}$$

for $x \in c_{00}(\mathbb{N})$ and $i \in \mathbb{N}$.
Then T is bounded, in fact, $\|Tx\|_\infty \leq \|x\|_\infty$ for all $x \in c_{00}(\mathbb{N})$.
On the other hand, we note that if $\lambda \in \mathbb{C}$ and $x \in c_{00}(\mathbb{N})$, then

$$(T - \lambda)x(k) = (\frac{1}{k+1} - \lambda)x(k).$$

From this we see that $\sigma_p(T) = \{1, \frac{1}{2}, \frac{1}{3}, ...\}$. *And if* $\lambda \notin \{1, \frac{1}{2}, \frac{1}{3}, ...\}$, *then* $T - \lambda$ *is an linear isomorphism and its inverse is given by*

$$(T-\lambda)^{-1}x(k) = (\frac{1}{k+1} - \lambda)^{-1}x(k).$$

So, $(T-\lambda)^{-1}$ *is unbounded if* $\lambda = 0$ *and thus* $0 \in \sigma(T)$.
On the other hand, if $\lambda \neq 0$, *then* $(T-\lambda)^{-1}$ *is bounded. In fact, if* $\lambda = a + ib \neq 0$, *for* $a, b \in \mathbb{R}$, *then* $\eta := \min_k |\frac{1}{1+k} - a|^2 + |b|^2 > 0$ *because* $\lambda \notin \{1, \frac{1}{2}, \frac{1}{3}, ...\}$. *This gives*

$$\|(T-\lambda)^{-1}\| = \sup_{k\in\mathbb{N}} |(\frac{1}{k+1} - \lambda)^{-1}| < \eta^{-1} < \infty.$$

It can now be concluded that $\sigma(T) = \{1, \frac{1}{2}, \frac{1}{3}, ...\} \cup \{0\}$.

Proposition 14.5. *Let* E *be a Banach space and* $T \in B(E)$. *Then*

(i) : $I - T$ *is invertible in* $B(E)$ *whenever* $\|T\| < 1$.
(ii) : *If* $|\lambda| > \|T\|$, *then* $\lambda \notin \sigma(T)$.
(iii) : $\sigma(T)$ *is a compact subset of* $\mathbb{C}$.
(iv) : *If we let* $GL(E)$ *the set of all invertible elements in* $B(E)$, *then* $GL(E)$ *is an open subset of* $B(E)$ *with respect to the* $\|\cdot\|$*-topology.*

Proof. Notice that since $B(E)$ is complete, Part (i) clearly follows from the following equality immediately:

$$(I-T)(I + T + T^2 + \cdots\cdots + T^{N-1}) = I - T^N$$

for all $N \in \mathbb{N}$.
For Part (ii), if $|\lambda| > \|T\|$, then by Part (i) , we see that $I - \frac{1}{\lambda}T$ is invertible and so is $\lambda I - T$. This implies $\lambda \notin \sigma(T)$.
For Part (iii), since $\sigma(T)$ is bounded by Part (ii), it needs to show that $\sigma(T)$ is closed.
Let $c \in \mathbb{C} \setminus \sigma(T)$. It needs to find $r > 0$ such that $\mu \notin \sigma(T)$ as $|\mu - c| < r$. Note that since $T - c$ is invertible, then for $\mu \in \mathbb{C}$, we have $T - \mu = (T-c) - (\mu - c) = (T-c)(I - (\mu-c)(T-c)^{-1})$. Therefore, if $\|(\mu - c)(T-c)^{-1})\| < 1$, then $T - \mu$ is invertible by Part (i). So if we take $0 < r < \dfrac{1}{\|(T-c)^{-1}\|}$, then r is as desired, that is, $B(c, r) \subseteq \mathbb{C} \setminus \sigma(T)$. Hence $\sigma(T)$ is closed.
For the last assertion, let $T \in GL(E)$. Notice that for any $S \in B(E)$, we have $\|T - S\| \leq \|T\|\|I - T^{-1}S\|$. So if $\|S\| < \dfrac{1}{\|T^{-1}\|}$, then $T - S$ is invertible by Part (i). Therefore we have $B(T, \frac{1}{\|T^{-1}\|}) \subseteq GL(E)$.
The proof is finished. □

Corollary 14.6. *If* U *is a unitary operator on a Hilbert space* X, *then* $\sigma(U) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Proof. Since $\|U\| = 1$, we have $\sigma(U) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ by Proposition 14.5(ii).
Now if $|\lambda| < 1$, then $\|\lambda U^*\| < 1$. By using Proposition 14.5 again, we have $I - \lambda U^*$ is invertible. This implies that $U - \lambda = U(I - \lambda U^*)$ is also invertible and thus $\lambda \notin \sigma(U)$. □

Example 14.7. *Let* $E = \ell^2(\mathbb{N})$ *and* $D \in B(E)$ *be the right unilateral shift operator as in Example13.8. Recall that* $Dx(k) := x(k-1)$ *for* $i \in \mathbb{N}$ *and* $x(-1) := 0$. *Then* $\sigma_p(D) = \emptyset$ *and* $\sigma(D) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.
We first claim that $\sigma_p(D) = \emptyset$.
Suppose that $\lambda \in \mathbb{C}$ *and* $x \in \ell^2(\mathbb{N})$ *satisfy the equation* $Dx = \lambda x$. *Then by the definition of* D, *we have*

$$x(k-1) = \lambda x(k) \qquad \cdots\cdots\cdots (*)$$

for all $k \in \mathbb{N}$.
If $\lambda \neq 0$, *then we have* $x(k) = \lambda^{-1}x_{k-1}$ *for all* $i \in \mathbb{N}$. *Since* $x(-1) = 0$, *this forces* $x(k) = 0$ *for all* i, *that is* $x = 0$ *in* $\ell^2(\mathbb{N})$.
On the other hand if $\lambda = 0$, *the Eq.*$(*)$ *gives* $x(k-1) = 0$ *for all* k *and so* $x = 0$ *again.*
Therefore $\sigma_p(D) = \emptyset$.
Finally, we are going to show $\sigma(D) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.
Note that since D *is an isometry,* $\|D\| = 1$. *Proposition 14.5 tells us that*

$$\sigma(D) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}.$$

Notice that since $\sigma_p(D)$ *is empty, it suffices to show that* $D - \mu$ *is not surjective for all* $\mu \in \mathbb{C}$ *with* $|\mu| \leq 1$.
Now suppose that there is $\lambda \in \mathbb{C}$ *with* $|\lambda| \leq 1$ *such that* $D - \lambda$ *is surjective.*
We consider the case when $|\lambda| = 1$ *first.*
Let $e_1 = (1, 0, 0, ...) \in \ell^2(\mathbb{N})$. *Then by the assumption, there is* $x \in \ell^2(\mathbb{N})$ *such that* $(D - \lambda)x = e_1$ *and thus* $Dx = \lambda x + e_1$. *This implies that*

$$x(k-1) = Dx(k) = \lambda x(k) + e_1(k)$$

for all $k \in \mathbb{N}$. *From this we have* $x(0) = -\lambda^{-1}$ *and* $x(k) = -\lambda^{-k}x(0)$ *for all* $k \geq 1$ *because since* $e_1(0) = 1$ *and* $e_1(k) = 0$ *for all* $k \geq 1$. *Also since* $|\lambda| = 1$, *it turns out that* $|x(0)| = |x(k)|$ *for all* $k \geq 1$. *As* $x \in \ell^2(\mathbb{N})$, *this forces* $x = 0$. *However, it is absurd because* $Dx = \lambda x + e_1$.
Now we consider the case when $|\lambda| < 1$.
Notice that by Proposition 13.13, we have

$$\overline{im(D-\lambda)}^{\perp} = \ker(D-\lambda)^* = \ker(D^* - \overline{\lambda}).$$

Thus if $D - \lambda$ *is surjective, we have* $\ker(D^* - \overline{\lambda}) = (0)$ *and hence* $\overline{\lambda} \notin \sigma_p(D^*)$.
Notice that the adjoint D^* *of* D *is given by the left shift operator, that is,*

$$D^*x(k) = x(k+1) \qquad \cdots\cdots\cdots (**)$$

for all $k \in \mathbb{N}$.
Now when $D^*x = \mu x$ *for some* $\mu \in \mathbb{C}$ *and* $x \in \ell^2(\mathbb{N})$, *by using Eq.*$(**)$, *which is equivalent to saying that*

$$x(k+1) = \mu x(k)$$

for all $k \in \mathbb{N}$. *So as* $|\overline{\lambda}| = |\lambda| < 1$, *if we set* $x(0) = 1$ *and* $x(k+1) = \overline{\lambda}^k x(0)$ *for all* $k \geq 1$, *then* $x \in \ell^2(\mathbb{N})$ *and* $D^*x = \overline{\lambda}x$. *Hence* $\overline{\lambda} \in \sigma_p(D^*)$ *which leads to a contradiction.*
The proof is finished.

15. Spectral Theory II

Throughout this section, let H be a complex Hilbert space.

Lemma 15.1. *Let* $T \in B(H)$ *be a normal operator (recall that* $T^*T = TT^*$*). Then* T *is invertible in* $B(H)$ *if and only if there is* $c > 0$ *such that* $\|Tx\| \geq c\|x\|$ *for all* $x \in H$.

Proof. The necessary part is clear.
Now we are going to show the converse. We first to show the case when T is selfadjoint. It is clear that T is injective from the assumption. So by the Open Mapping Theorem, it remains to show that T is surjective.
In fact since $\ker T = \overline{imT^*}^{\perp}$ and $T = T^*$, we see that the image of T is dense in H.
Now if $y \in H$, then there is a sequence (x_n) in H such that $Tx_n \to y$. So (Tx_n) is a Cauchy sequence. From this and the assumption give us that (x_n) is also a Cauchy sequence. If x_n converges to $x \in H$, then $y = Tx$. Therefore the assertion is true when T is selfadjoint.

Now if T is normal, then we have $\|T^*x\| = \|Tx\| \geq c\|x\|$ for all $x \in H$ by Proposition 13.11(ii). Therefore, we have $\|T^*Tx\| \geq c\|Tx\| \geq c^2\|x\|$. Hence T^*T still satisfies the assumption. Notice that T^*T is selfadjoint. So we can apply the previous case to know that T^*T is invertible. This implies that T is also invertible because $T^*T = TT^*$.
The proof is finished. □

Definition 15.2. *Let $T \in B(X)$. We say that T is positive, write $T \geq 0$, if $(Tx, x) \geq 0$ for all $x \in H$.*

Remark 15.3. *It is clear that a positive operator is selfadjoint by Proposition 13.12 at once. In particular, all projections are positive.*

Proposition 15.4. *Let $T \in B(H)$. We have*

(i) : If $T \geq 0$, then $T + I$ is invertible.
(ii) : If T is self-adjoint, then $\sigma(T) \subseteq \mathbb{R}$. In particular, when $T \geq 0$, we have $\sigma(T) \subseteq [0, \infty)$.

Proof. For Part (i), we assume that $T \geq 0$. This implies that

$$\|(I + T)x\|^2 = \|x\|^2 + \|Tx\|^2 + 2(Tx, x) \geq \|x\|^2$$

for all $x \in H$. So the invertibility of $I + T$ follows from Lemma 15.1.
For Part (ii), we first claim that $T + i$ is invertible. Indeed, it follows from $(T + i)^*(T + i) = T^2 + I$ and Part (i) immediately.
Now if $\lambda = a + ib \in \sigma(T)$ where $a, b \in \mathbb{R}$ with $b \neq 0$, then $T - \lambda = -b(\frac{-1}{b}(T - a) + i)$ is invertible because $\frac{-1}{b}(T - a)$ is selfadjoint.
Finally we are going to show $\sigma(T) \subseteq [0, \infty)$ when $T \geq 0$. Notice that since $\sigma(T) \subseteq \mathbb{R}$, it suffices to show that $T - c$ is invertible if $c < 0$. Indeed, if $c < 0$, then we see that $T - c = -c(I + (\frac{-1}{c}T))$ is invertible by the previous assertion because $\frac{-1}{c}T \geq 0$.
The proof is finished. □

Remark 15.5. *In Proposition 15.4, we have shown that if T is selfadjoint, then $\sigma(T) \subseteq \mathbb{R}$. However, the converse does not hold. For example, consider $H = \mathbb{C}^2$ and*

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Theorem 15.6. *Let $T \in B(H)$ be a selfadjoint operator. Put*

$$M(T) := \sup_{\|x\|=1} (Tx, x) \quad \text{and} \quad m(T) = \inf_{\|x\|=1} (Tx, x).$$

For convenience, we also write $M = M(T)$ and $m = m(T)$ if there is no confusion. Then we have

(i) : $\|T\| = \max\{|m|, |M|\}$.
(ii) : $\{m, M\} \subseteq \sigma(T)$.
(iii) : $\sigma(T) \subseteq [m, M]$.

Proof. Notice that m and M are defined because (Tx, x) is real for all $x \in H$ by Proposition 13.12 (ii). Also Part(i) can be obtained by using Lemma 13.12 (ii) again.
For Part (ii), we first claim that $M \in \sigma(T)$ if $T \geq 0$. Notice that $0 \leq m \leq M = \|T\|$ in this case by Lemma 13.12. Then there is a sequence (x_n) in H with $\|x_n\| = 1$ for all n such that $(Tx_n, x_n) \to M = \|T\|$. Then we have

$$\|(T - M)x_n\|^2 = \|Tx_n\|^2 + M^2\|x_n\|^2 - 2M(Tx_n, x_n) \leq \|T\|^2 + M^2 - 2M(Tx_n, x_n) \to 0.$$

So by Lemma 15.1 we have shown that $T - M$ is not invertible and hence $M \in \sigma(T)$ if $T \geq 0$.
Now for any selfadjoint operator T if we consider $T - m$, then $T - m \geq 0$. Thus we have $M - m = M(T - m) \in \sigma(T - m)$ by the previous case. It is clear that $\sigma(T - c) = \sigma(T) - c$ for all $c \in \mathbb{C}$. Therefore we have $M \in \sigma(T)$ for any self-adjoint operator.
We are now claiming that $m(T) \in \sigma(T)$. Notice that $M(-T) = -m(T)$. So we have $-m(T) \in \sigma(-T)$. It is clear that $\sigma(-T) = -\sigma(T)$. Then $m(T) \in \sigma(T)$.
Finally, we are going to show $\sigma(T) \subseteq [m, M]$.
Indeed, since $T - m \geq 0$, then by Proposition 15.4, we have $\sigma(T) - m = \sigma(T - m) \subseteq [0, \infty)$. This gives $\sigma(T) \subseteq [m, \infty)$.
On the other hand, similarly, we consider $M - T \geq 0$. Then we get $M - \sigma(T) = \sigma(M - T) \subseteq [0, \infty)$. This implies that $\sigma(T) \subseteq (-\infty, M]$. The proof is finished. □

16. Compact operators on a Hilbert space

Throughout this section, let H be a complex Hilbert space.

Definition 16.1. *A linear operator $T : H \to H$ is said to be compact if for every bounded sequence (x_n) in H, $(T(x_n))$ has a norm convergent subsequence.*
Write $K(H)$ for the set of all compact operators on H and $K(H)_{sa}$ for the set of all compact selfadjoint operators.

Remark 16.2. *Let U be the closed unit ball of H. It is clear that T is compact if and only if the norm closure $\overline{T(U)}$ is a compact subset of H. Thus if T is compact, then T is bounded automatically because every compact set is bounded.*
Also it is clear that if T has finite rank, that is $\dim\ imT < \infty$, then T must be compact because every closed and bounded subset of a finite dimensional normed space is equivalent to it is compact.

Example 16.3. *The identity operator $I : H \to H$ is compact if and only if $\dim H < \infty$.*

Example 16.4. *Let $H = \ell^2(\{1, 2...\})$. Define $Tx(k) := \frac{x(k)}{k}$ for $k = 1, 2....$ Then T is compact.*
In fact, if we let (x_n) be a bounded sequence in ℓ^2, then by the diagonal argument, we can find a subsequence $y_m := Tx_m$ of Tx_n such that $\lim_{m\to\infty} y_m(k) = y(k)$ exists for all $k = 1, 2...$ Let $L := \sup_n \|x_n\|_2^2$. Since $|y_m(k)|^2 \leq \frac{L}{k^2}$ for all m, k, we have $y \in \ell^2$. Now let $\varepsilon > 0$. Then one can find a positive integer N such that $\sum_{k\geq N} 4L/k^2 < \varepsilon$. So we have

$$\sum_{k\geq N} |y_m(k) - y(k)|^2 < \sum_{k\geq N} \frac{4L}{k^2} < \varepsilon$$

for all m. On the other hand, since $\lim_{m\to\infty} y_m(k) = y(k)$ for all k, we can choose a positive integer M such that

$$\sum_{k=1}^{N-1} |y_m(k) - y(k)|^2 < \varepsilon$$

for all $m \geq M$. Finally, we have $\|y_m - y\|_2^2 < 2\varepsilon$ for all $m \geq M$.

Theorem 16.5. *Let $T \in B(H)$. Then T is compact if and only if T maps every weakly convergent sequence in H to a norm convergent sequence.*

Proof. We first assume that $T \in K(H)$. Let (x_n) be a weakly convergent sequence in H. Since H is reflexive, (x_n) is bounded by the Uniform Boundedness Theorem. So we can find a subsequence (x_j) of (x_n) such that (Tx_j) is norm convergent. Let $y := \lim_j Tx_j$. We claim that $y = \lim_n Tx_n$. Suppose not. Then by the compactness of T again, we can find a subsequence (x_i) of (x_n) such that Tx_i converges to y' with $y \neq y'$. Thus there is $z \in H$ such that $(y, z) \neq (y', z)$. On the other hand, if we let x be the weakly limit of (x_n), then $(x_n, w) \to (x, w)$ for all $w \in H$. So we have

$$(y, z) = \lim_j (Tx_j, z) = \lim_j (x_j, T^*(z)) = (x, T^*z) = (Tx, z).$$

Similarly, we also have $(y', z) = (Tx, z)$ and hence $(y, z) = (y', z)$ that contradicts to the choice of z.
For the converse, let (x_n) be a bounded sequence. Then by Theorem 12.12, (x_n) has a weakly convergent subsequence. Thus $T(x_n)$ has a norm convergent subsequence by the assumption at once. So T is compact. □

Proposition 16.6. *Let $S, T \in K(H)$. Then we have*
(i) : $\alpha S + \beta T \in K(H)$ for all $\alpha, \beta \in \mathbb{C}$;
(ii) : TQ and $QT \in K(H)$ for all Q in $B(H)$;
(iii) : $T^* \in K(H)$.
Moreover $K(H)$ is normed closed in $B(H)$.
Hence $K(H)$ is a closed $*$-ideal of $B(H)$.

Proof. (i) and (ii) are clear.
For property (iii), let (x_n) be a bounded sequence. Then (T^*x_n) is also bounded. So TT^*x_n has a convergent subsequence $TT^*x_{n_k}$ by the compactness of T. Notice that we have

$$\|T^*x_{n_k} - T^*x_{n_l}\|^2 = (TT^*(x_{n_k} - x_{n_l}), x_{n_k} - x_{n_l})$$

for all n_k, n_l. This implies that $(T^*x_{n_k})$ is a Cauchy sequence and thus is convergent since (x_{n_k}) is bounded.
Finally we are going to show $K(H)$ is closed. Let (T_m) be a sequence in $K(H)$ such that $T_m \to T$ in norm. Let (x_n) be a bounded sequence in H. Then by the diagonal argument there is a subsequence (x_{n_k}) of (x_n) such that $\lim\limits_k T_m x_{n_k}$ exists for all m. Now let $\varepsilon > 0$. Since $\lim_m T_m = T$, there is a positive integer N such that $\|T - T_N\| < \varepsilon$. On the other hand, there is a positive integer K such that $\|T_N x_{n_k} - T_N x_{n_{k'}}\| < \varepsilon$ for all $k, k' \geq K$. So we can now have

$$\|Tx_{n_k} - Tx_{n_{k'}}\| \leq \|Tx_{n_k} - T_N x_{n_k}\| + \|T_N x_{n_k} - T_N x_{n_{k'}}\| + \|T_N x_{n_{k'}} - Tx_{n_{k'}}\| \leq (2L + 1)\varepsilon$$

for all $k, k' \geq K$ where $L := \sup_n \|x_n\|$. Thus $\lim_k Tx_{n_k}$ exists. It can now be concluded that $T \in K(H)$. The proof is finished. □

Corollary 16.7. *Let $T \in K(H)$. If $\dim H = \infty$, then $0 \in \sigma(T)$.*

Proof. Suppose that $0 \notin \sigma(T)$. Then T^{-1} exists in $B(H)$. Proposition 16.1 gives $I = TT^{-1} \in K(H)$. This implies $\dim H < \infty$. □

Proposition 16.8. *Let $T \in K(H)$ and let $c \in \mathbb{C}$ with $c \neq 0$. Then $T - c$ has a closed range.*

Proof. Notice that since $\frac{1}{c}T \in K(H)$, so if we consider $\frac{1}{c}T - I$, we may assume that $c = 1$. Let $S = T - I$. Let x_n be a sequence in H such that $Sx_n \to x \in H$ in norm. By considering the orthogonal decomposition $H = \ker S \oplus (\ker S)^\perp$, we write $x_n = y_n \oplus z_n$ for $y_n \in \ker S$ and $z_n \in (\ker S)^\perp$. We first claim that (z_n) is bounded. Suppose not. By considering a subsequence of (z_n), we may assume that we may assume that $\|z_n\| \to \infty$. Put $v_n := \dfrac{z_n}{\|z_n\|} \in (\ker S)^\perp$.

Since $Sz_n = Sx_n \to x$, we have $Sv_n \to 0$. On the other hand, since T is compact, and (v_n) is bounded, by passing a subsequence of (v_n), we may also assume that $Tv_n \to w$. Since $S = T - I$, $v_n = Tv_n - Sv_n \to w - 0 = w \in (\ker S)^\perp$. Also from this we have $Sv_n \to Sw$. On the other hand, we have $Sw = \lim_n Sv_n = \lim_n Tv_n - \lim_n v_n = w - w = 0$. So $w \in \ker S \cap (\ker S)^\perp$. It follows that $w = 0$. However, since $v_n \to w$ and $\|v_n\| = 1$ for all n. It leads to a contradiction. So (z_n) is bounded.

Finally we are going to show that $x \in imS$. Now since (z_n) is bounded, (Tz_n) has a convergent subsequence (Tz_{n_k}). Let $\lim_k Tz_{n_k} = z$. Then we have

$$z_{n_k} = Sz_{n_k} - Tz_{n_k} = Sx_{n_k} - Tz_{n_k} \to x - z.$$

It follows that $x = \lim_k Sx_{n_k} = \lim_k Sz_{n_k} = S(x - z) \in imS$. The proof is finished. □

Theorem 16.9. Fredholm Alternative Theorem : *Let $T \in K(H)_{sa}$ and let $0 \neq \lambda \in \mathbb{C}$. Then $T - \lambda$ is injective if and only if $T - \lambda$ is surjective.*

Proof. Since T is selfadjoint, $\sigma(T) \subseteq \mathbb{R}$. So if $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $T - \lambda$ is invertible. So the result holds automatically.

Now consider the case $\lambda \in \mathbb{R} \setminus \{0\}$.

Then $T - \lambda$ is also selfadjoint. From this and Proposition 13.13, we have $\ker(T - \lambda) = (im(T - \lambda))^\perp$ and $(\ker(T - \lambda))^\perp = \overline{im(T - \lambda)}$.

So the proof is finished by using Proposition 16.8 immediately. □

Corollary 16.10. *Let $T \in K(H)_{sa}$. Then we have $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$. Consequently if the values $m(T)$ and $M(T)$ which are defined in Theorem 15.6 are non-zero, then both are the eigenvalues of T and $\|T\| = \max\limits_{\lambda \in \sigma_p(T)} |\lambda|$.*

Proof. It follows from the Fredholm Alternative Theorem at once. This together with Theorem 15.6 imply the last assertion. □

Example 16.11. *Let $T \in B(\ell^2)$ be defined as in Example 16.4. We have shown that $T \in K(\ell^2)$ and it is clear that T is selfadjoint. Then by Corollary 16.10 and Corollary 16.7, we see that $\sigma(T) = \{0, 1, \frac{1}{2}, \frac{1}{3}, ...\}$.*

Lemma 16.12. *Let $T \in K(H)_{sa}$ and let $E_\lambda := \{x \in H : Tx = \lambda x\}$ for $\lambda \in \sigma(T) \setminus \{0\}$, that is the eigenspace of T corresponding to λ. If we fix $\mu \in \sigma(T) \setminus \{0\}$ and put $I_\mu := \{\lambda \in \sigma(T) : |\lambda| = |\mu|\}$, then we have*

$$\dim \bigoplus_{\lambda \in I_\mu} E_\lambda < \infty.$$

Proof. We first notice that $\dim E_\lambda < \infty$ for all $\lambda \in \sigma_p(T) \setminus \{0\}$ because the restriction $T|E_\lambda$ is also a compact operator on E_λ.

On the other hand, since T is selfajoint, we also have $E_\lambda \perp E_{\lambda'}$ for $\lambda, \lambda' \in \sigma_p(T)$ with $\lambda \neq \lambda'$. Let $V := \bigoplus_{\lambda \in I_\mu} E_\lambda$. Suppose that $\dim V = \infty$. Then $|I_\mu| = \infty$. So, we can find an infinite sequence in I_μ such that $\lambda_m \neq \lambda_n$ for $m \neq n$. Now choose $v_n \in E_{\lambda_n}$ with $\|v_n\| = 1$ for each λ_n. Then $v_n \perp v_m$ for $n \neq m$. This implies that $\|Tv_n - Tv_m\|^2 = |\lambda_n|^2 + |\lambda_m|^2 = 2|\mu|^2 > 0$ for $m \neq n$. So (Tv_n) has no convergent subsequences which contradicts to T being compact. □

Theorem 16.13. *Let $T \in K(H)_{sa}$. And suppose that $\dim H = \infty$. Then $\sigma(T) = \{\lambda_1, \lambda_2, ...\} \cup \{0\}$, where (λ_n) is a sequence of real numbers with $\lambda_n \neq \lambda_m$ for $m \neq n$ and $|\lambda_n| \downarrow 0$.*

Proof. Note that since $\|T\| = \max(|M(T)|, |m(T)|)$ and $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$. So by Corollary 16.10, there is $|\lambda_1| = \max\limits_{\lambda\in\sigma_p(T)} |\lambda| = \|T\|$. Since $\dim E_{\lambda_1} < \infty$, then $E_{\lambda_1}^{\perp} \neq 0$. Then by considering the restriction of $T_2 := T|E_{\lambda_1}^{\perp} \neq 0$, there is $|\lambda_2| = \max_{\lambda\in\sigma_p(T_2)} |\lambda| = \|T_2\|$. Notice that $\lambda_2 \in \sigma_p(T)$ and $|\lambda_2| \leq |\lambda_1|$ because $\|T_2\| \leq \|T\|$. To repeat the same step, we can get a sequence (λ_n) such that $(|\lambda_n|)$ is decreasing.
Now we claim that $\lim_n |\lambda_n| = 0$.
Otherwise, there is $\eta > 0$ such that $|\lambda_n| \geq \eta$ for all n. If we let $v_n \in E_{\lambda_n}$ with $\|v_n\| = 1$ for all n. Notice that since $\dim H = \infty$ and $\dim E_\lambda < \infty$, for any $\lambda \in \sigma_p(T) \setminus \{0\}$, there are infinite many λ_n's. Then $w_n := \frac{1}{|\lambda_n|} v_n$ is a bounded sequence and $\|Tw_n - Tw_n\|^2 = \|v_n - v_m\|^2 = 2$ for $m \neq n$. This is a contradiction since T is compact. So $\lim_n |\lambda_n| = 0$.
Finally we need to check $\sigma(T) = \{\lambda_1, \lambda_2, ...\} \cup \{0\}$.
In fact, let $\mu \in \sigma_p(T)$. Since $|\lambda_n| \downarrow 0$, we can find a subsequence $n_1 < n_2 < ....$ of positive integers such that

$$|\lambda_1| = ... = |\lambda_{n_1}| > |\lambda_{n_1+1}| = ... = |\lambda_{n_2}| > |\lambda_{n_2+1}| = ... = |\lambda_{n_3}| > |\lambda_{n_3+1}| = ...$$

Then we can choose N such that $|\lambda_{n_N+1}| < |\mu| \leq |\lambda_{n_N}|$. Notice that by the construction of λ_n's implies $\mu = \lambda_j$ for some $n_{N-1} + 1 \leq j \leq n_N$.
The proof is finished. □

Theorem 16.14. *Let $T \in K(H)_{sa}$ and let (λ_n) be given as in Theorem 16.13. For each $\lambda \in \sigma_p(T) \setminus \{0\}$, put $d(\lambda) := \dim E_\lambda < \infty$. Let $\{e_{\lambda,i} : i = 1, ..., d(\lambda)\}$ be an orthonormal base for E_λ. Then we have the following orthogonal decomposition:*

$$H = \ker T \oplus \bigoplus_{n=1}^{\infty} E_{\lambda_n}. \tag{16.1}$$

Moreover $\mathcal{B} := \{e_{\lambda,i} : \lambda \in \sigma_p(T) \setminus \{0\}; i = 1, .., d(\lambda)\}$ forms an orthonormal base of $\overline{T(H)}$.
Also the series $\sum\limits_{n=1}^{\infty} \lambda_n P_n$ norm converges to T, where P_n is the orthogonal projection from H onto E_{λ_n}, that is, $P_n(x) := \sum\limits_{i=1}^{d(\lambda_n)} (x, e_{\lambda_n,i}) e_{\lambda_n,i}$, for $x \in H$.

Proof. Put $E = \bigoplus_{n=1}^{\infty} E_{\lambda_n}$. It is clear that $\ker T \subseteq E^{\perp}$. On the other hand, if the restriction $T_0 := T|E^{\perp} \neq 0$, then there exists an non-zero element $\mu \in \sigma_p(T_0) \subseteq \sigma_p(T)$ because $T_0 \in K(E^{\perp})$. It is absurd because $\mu \neq \frac{1}{\lambda_i}$ for all i. So $T|E^{\perp} = 0$ and hence $E^{\perp} \subseteq \ker T$. So we have the decomposition (16.1). And from this we see that the family $\mathcal{B}$ forms an orthonormal base of $(\ker T)^{\perp}$. On the other, we have $(\ker T)^{\perp} = \overline{imT^*} = \overline{imT}$. Therefore, $\mathcal{B}$ is an orthonormal base for $\overline{T(H)}$ as desired.
For the last assertion, it needs to show that the series $\sum_{n=1}^{\infty} \lambda_n P_n$ converges to T in norm. Notice that if we put $S_m := \sum_{n=1}^{m} \lambda_n P_n$, then by the decomposition (16.1), $\lim\limits_{m\to\infty} S_m x = Tx$ for all $x \in H$. So it suffices to show that $(S_m)_{m=1}^{\infty}$ is a Cauchy sequence in $B(H)$. In fact we have

$$\|\lambda_{m+1} P_{m+1} + \cdots\cdots + \lambda_{m+p} P_{m+p}\| = |\lambda_{m+1}|$$

for all $m, p \in \mathbb{N}$ because $E_{\lambda_n} \perp E_{\lambda_m}$ for $m \neq n$ and $|\lambda_n|$ is decreasing. This gives that (S_n) is a Cauchy sequence since $|\lambda_n| \downarrow 0$. The proof is finished. □

Corollary 16.15. *$T \in K(H)$ if only if T can be approximated by finite rank operators.*

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